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AIR FORCE INSTITUTE OF TECHNOLOGY  
Department of Mechanical Engineering  
LECTURE NOTES  
Thermodynamics of Gas Flow  
ME 257

160-P

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July 1956

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# SYMBOLS

# or lbf pound force

slug  $\equiv \frac{\text{\#-sec}^2}{\text{ft}}$

h enthalpy,  $\frac{\text{energy}}{\text{mass}}$   $\frac{\text{ft \#}}{\text{slug}}$  or  $\frac{\text{ft}^2}{\text{sec}^2}$

V velocity,  $\frac{\text{length}}{\text{time}}$  ft/sec

v specific volume  $\frac{\text{vol}}{\text{mass}}$   $\frac{\text{ft}^3}{\text{slug}}$

Q heat,  $\frac{\text{energy}}{\text{mass}}$   $\frac{\text{ft \#}}{\text{slug}}$

W work,  $\frac{\text{energy}}{\text{mass}}$   $\frac{\text{ft \#}}{\text{slug}}$

$c_p(c_v)$  specific heat at constant pressure (volume),  $\frac{\text{energy}}{\text{mass-temp}}$   $\frac{\text{ft \#}}{\text{slug } ^\circ\text{R}}$

T absolute temperature,  $^\circ\text{R}$

P pressure,  $\frac{\text{force}}{\text{length squared}}$   $\frac{\text{\#}}{\text{ft}^2}$

$\rho$  mass density,  $\frac{\text{mass}}{\text{volume}}$   $\frac{\text{slugs}}{\text{ft}^3}$

k ratio of specific heats,  $c_p/c_v$

R gas constant,  $\frac{\text{energy}}{\text{mass-temp}}$   $\frac{\text{ft \#}}{\text{slug } ^\circ\text{R}}$

w mass rate of flow,  $\frac{\text{mass}}{\text{time}}$   $\frac{\text{slugs}}{\text{sec.}}$

A area,  $\text{ft}^2$

M Mach number

a speed of sound ft/sec.

$M^* \equiv V/a^* = V/V^*$

F impulse function,  $p A + \rho A V^2$ , force - #

$\mathcal{F}$  force of fluid on duct, #

$\gamma$  specific weight,  $\frac{\text{force}}{\text{volume}}$   $\frac{\text{lbf}}{\text{ft}^3}$

Subscript 0 designates total or stagnation value of property.

Superscript \* designates value of property at point in flow where mach number is one. (Exception:  $M^*$  which is defined above).

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## Lecture Note 1

INTRODUCTION

These notes on Thermodynamics of Gas Flow will be concerned with a portion of that branch of engineering study called fluid mechanics. In order to become oriented with regard to the realm of fluid mechanics to be covered herein, the different realms of fluid mechanics are listed below:

(a) Acoustics. The fluid velocities are extremely small compared with the velocity of sound, and the variations in pressure, temperature, and density are also very small.

(b) Meteorology. The fluid velocities are extremely small compared with the velocity of sound, but the variations in pressure, temperature, and density are of significant magnitude.

(c) Incompressible Fluid Mechanics. The fluid velocities are small compared with the velocity of sound; the variations in temperature and density are small, but the variation in pressure may be significant. It may be shown that the error produced in the computation of pressure variations by neglecting density changes (compressibility) is of the order of one-fourth the square of the ratio of the stream velocity to the sound velocity; thus, this ratio may be as great as 0.2 (corresponding to a velocity of about 200 ft/sec for air at normal atmospheric temperature) before the computed error in the pressure variation exceeds one per cent. For many problems in the flow of gases, therefore, the flow may with little error be treated as incompressible.

(d) Compressible Fluid Mechanics. The fluid velocities are appreciable compared with the velocity of sound, and the variations in pressure, temperature, and density are all of significant magnitude.

The latter realm of fluid mechanics, often called Gas Dynamics, is the principal subject of these notes. Further, the study herein will be restricted almost entirely to that of one-dimensional flow.

Review of Basic Laws

Since the study of fluid flow, no matter how complicated, is based on the fundamental laws of conservation of mass, Newton's 2nd Law of Motion, the 1st Law of Thermodynamics, and the 2nd Law of Thermodynamics, these will be our tools of analysis. It is quite proper, therefore, to begin our study with a review and clarification of these laws as applied to one-dimensional fluid flow problems.

Consider in turn then the laws listed below:

<div>Applied to →</div> <div>Law ↓</div>	<u>System</u> - An arbitrary collection of matter of fixed identity.	<u>Control volume</u> - An arbitrary volume at a <u>fixed location in space</u> .
(1) Conservation of mass	$m = \text{constant}$ $\frac{dm}{dt} = 0$	
(2) Newton's 2nd Law of Motion	$F = ma = \frac{mdv}{dt}$ $F = \frac{d(mv)}{dt}$	
(3) 1st Law of Thermodynamics	$\Delta E = Q - W$	
(4) 2nd Law of Thermodynamics	$ds = \left(\frac{dq}{T}\right)$ $ds \geq dq/T$	

Each of these laws is stated in the first instance for a mass of fixed identity and the mathematical statement of the laws as given above apply to a mass of fixed identity or to a system\*. In fluid flow problems it is useful and convenient to have a mathematical statement of these laws as they apply to an arbitrary volume at a fixed location in space or to a control volume\*\*. We desire therefore, to develop such expressions and to complete the block diagram of the fundamental laws in succeeding notes.

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\* System is defined as an arbitrary collection of matter of fixed identity.

\*\* Control volume is defined as an arbitrary volume at a fixed location in space.



## Lecture Note 2

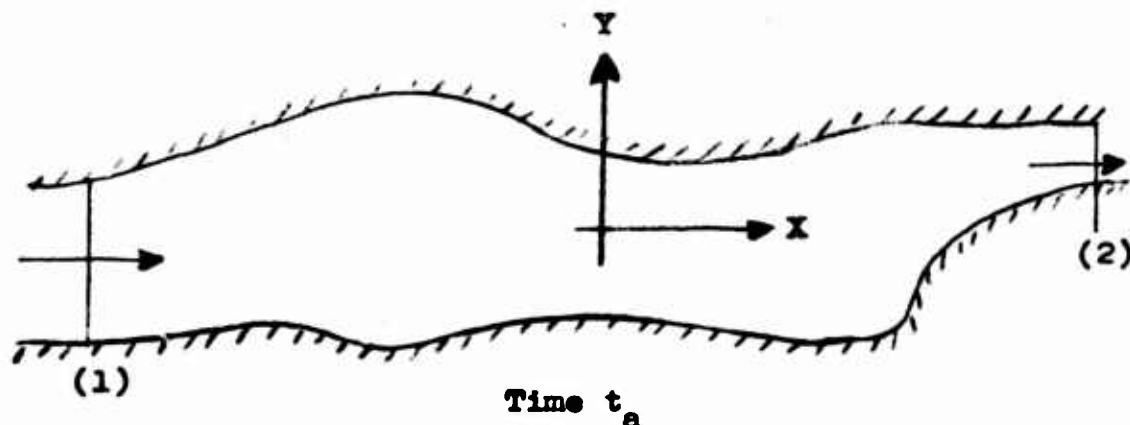
CONSERVATION OF MASS

The Law of Conservation of Mass states that mass can neither be created nor destroyed. Thus if we consider a quantity of matter of fixed identity and of mass  $m$  we can write for this system that the mass remains constant or does not vary with time and

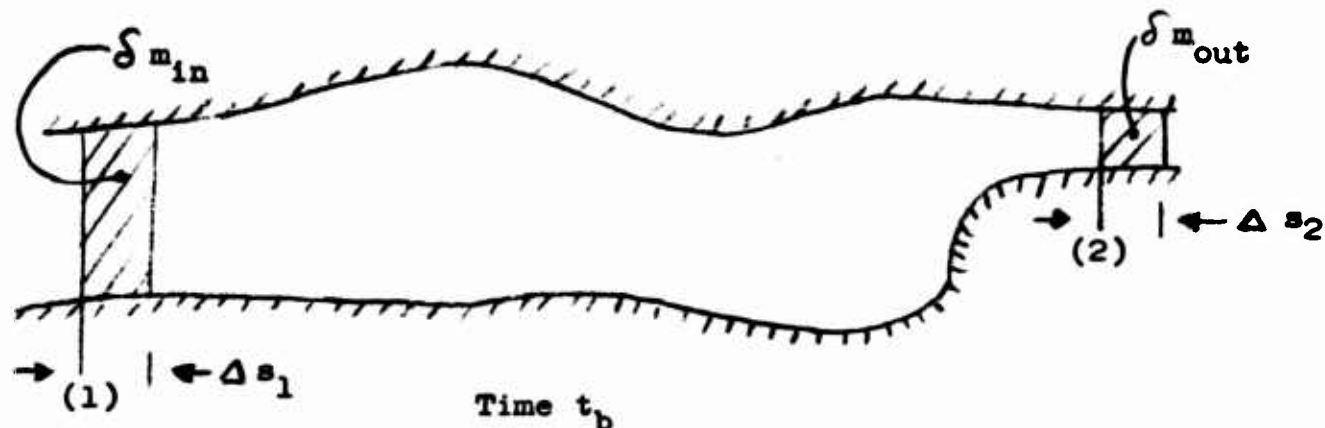
$$\frac{dm}{dt} = 0$$

Now consider using this equation to obtain an expression applicable to the flow of fluid through a control volume.

In the figure herewith fluid is flowing through a duct. Mark out a region bounded by the duct walls and sections (1) and (2) and designate this region as the control volume. We desire to obtain an expression for the derivative  $dm/dt$  when applied to the mass system of fluid which at time



Mass system contained within control volume



Mass system not coincident with control volume

$t_a$  is contained in the control volume. Recall that by definition

$$\frac{dm}{dt} = \lim_{\Delta t \rightarrow 0} \left( \frac{m_{tb} - m_{ta}}{\Delta t} \right)$$

where  $\Delta t = t_b - t_a$

$m_{t_a}$  = mass system at time  $t_a$

$m_{t_b}$  = mass system at time  $t_b$

Let the mass contained within the control volume at any instant of time be designated as  $\bar{m}$ . Notice that as time progresses  $\bar{m}$  identifies masses of different identity. This ( $\bar{m}$ ) is not to be confused with  $m$ ,  $m_{t_a}$ , or  $m_{t_b}$  which refer to a mass of fixed identity.\*

In order to evaluate the time derivative of  $m$  we notice that at time  $t_a$   $m$  and  $\bar{m}$  are identical so that

$$m_{t_a} = \bar{m}_{t_a}$$

On the other hand at time  $t_b$  our mass system is not completely bounded by the control surface. A small portion of the system, denoted by  $\delta m_{out}$ , has moved out of the control volume while a much larger portion is still in, and occupies most of, the control volume. Also during the time  $\Delta t$  a mass  $\delta m_{in}$ , foreign to our system, has flowed into the control volume as indicated in the figure. We note, therefore, that at time  $t_b$  our mass system consists of the mass in the control region less the foreign mass  $\delta m_{in}$  and plus the mass  $\delta m_{out}$ . This gives

$$m_{t_b} = \bar{m}_{t_b} - \delta m_{in} + \delta m_{out}$$

Using these we can write

$$m_{t_b} - m_{t_a} = \bar{m}_{t_b} - \bar{m}_{t_a} + \delta m_{out} - \delta m_{in}$$

and

$$\frac{dm}{dt} = \lim_{\Delta t \rightarrow 0} \left\{ \frac{\bar{m}_{t_b} - \bar{m}_{t_a}}{\Delta t} + \frac{\delta m_{out} - \delta m_{in}}{\Delta t} \right\}$$

---

\*By the Law of Conservation of Mass  $\frac{dm}{dt} = 0$  but it does not follow that  $\frac{d\bar{m}}{dt} = 0$ . This illustrates the point that the law applies to a collection of matter of fixed identity.



Now  $\delta m_{\text{out}} = \rho_2 A_2 \Delta s_2$

$$\delta m_{\text{in}} = \rho_1 A_1 \Delta s_1$$

since, by making  $\Delta t$  small enough, the density and cross-sectional area throughout  $\delta m$  are constant and equal to the value at station (1) or station (2). Substituting for  $\delta m$  and taking limits we obtain

$$\frac{dm}{dt} = \frac{dm}{dt}_{\text{c.v.}} + \rho_2 A_2 V_2 - \rho_1 A_1 V_1$$

which states that the rate of change of our mass of fixed identity equals the rate of change of the quantity of mass in the control volume plus the net outflow of mass from the control volume. Now  $dm/dt = 0$  so

$$\boxed{\left(\frac{dm}{dt}\right)_{\text{c.v.}} = \rho_1 A_1 V_1 - \rho_2 A_2 V_2} \quad (1)$$

which states that accumulation of mass in c.v. = mass inflow - mass outflow.

In case of steady flow  $\left(\frac{dm}{dt}\right)_{\text{c.v.}} = 0$  and

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \quad (1a)$$

which relation is known as the continuity equation. Thus we have developed a mathematical statement - eqn (1) - of mass conservation as applied to the flow of fluid through a control volume.

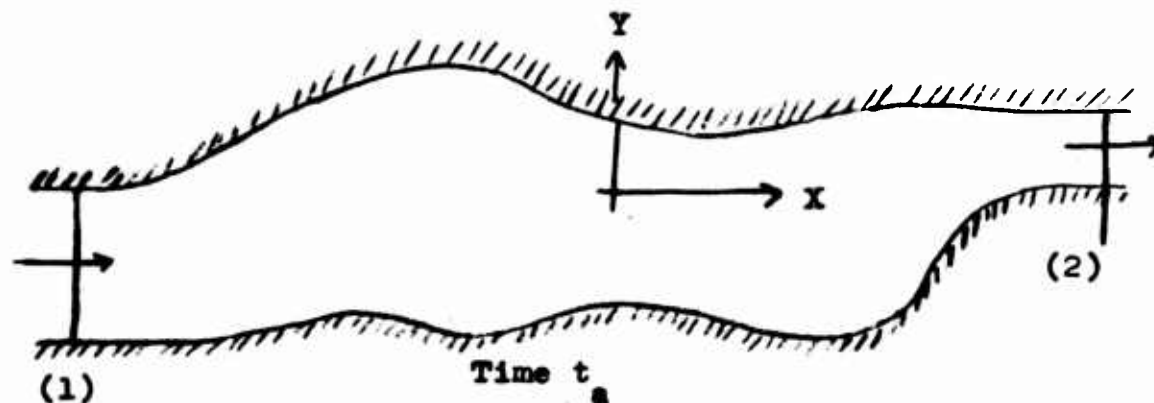
## Lecture Note 3

In the application of Newton's Second Law of Motion to fluid flow studies it is useful to have a mathematical statement of the law which will directly apply to flow through a control volume. Starting with the equation

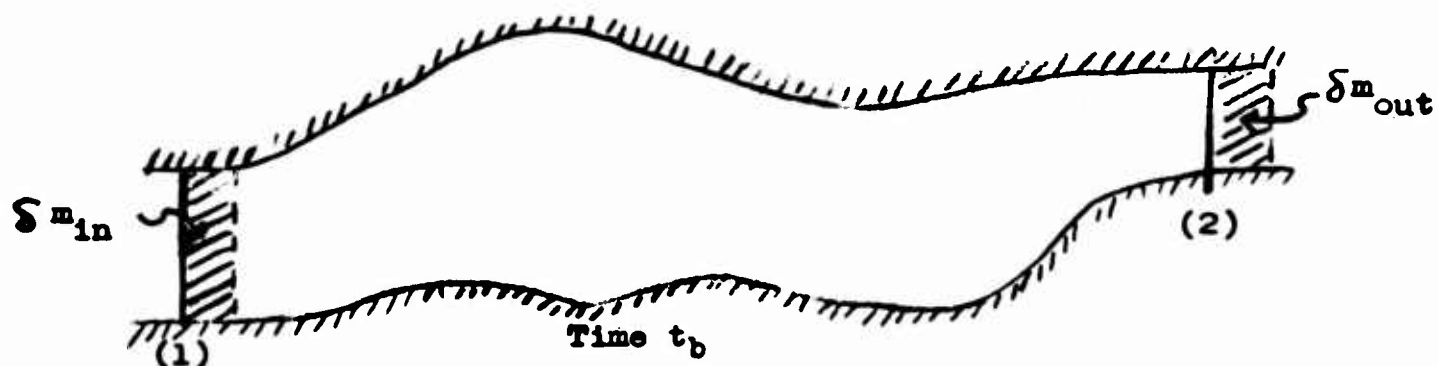
$$(\text{Force on mass system})_x \equiv \frac{d(\text{Momentum of mass system})_x}{dt}$$

we will in this note develop a "control volume expression of the 2nd Law of Motion" following a procedure completely analogous to that used in obtaining the continuity equation or "control volume expression of the Law of Mass Conservation". In the present case it will be necessary to evaluate the time derivative of the momentum,  $M$ , of a system in conjunction with flow of fluid through a control volume.\* (See footnote page 3.2)

Consider flow through the region (control volume) bounded by the duct walls and the sections (1) and (2) shown in the figure below. In this derivation we require that the stream properties at (1) and (2) be uniform across each respective section and that the velocities be in the same direction.



Mass system contained within control volume



Mass system not coincident with control volume

It is desired to evaluate the derivative

$$\frac{d(\text{momentum of mass})_x}{dt}$$

at the time  $t_a$  using

$$\frac{d(\text{momentum of mass})_x}{dt} = \lim_{\Delta t \rightarrow 0} \left( \frac{M_{t_b} - M_{t_a}}{\Delta t} \right)$$

where  $M = x\text{-momentum of mass system under consideration.}$

If we let  $\bar{M} = x\text{-momentum of fluid contained within control volume at any instant}$

$$\text{then } M_{t_b} = \bar{M}_{t_b} - v_1 \delta m_{in} + v_2 \delta m_{out}$$

$$M_{t_a} = \bar{M}_{t_a}$$

We obtain, therefore

$$dM/dt = \lim_{\Delta t \rightarrow 0} \left\{ \frac{(\bar{M}_{t_b} - v_1 \delta m_{in} + v_2 \delta m_{out}) - \bar{M}_{t_a}}{\Delta t} \right\}$$

$$\text{or } \lim_{\Delta t \rightarrow 0} \left\{ \left( \frac{\bar{M}_{t_b} - \bar{M}_{t_a}}{\Delta t} \right) + \left( \frac{v_2 \delta m_{out} - v_1 \delta m_{in}}{\Delta t} \right) \right\}$$

Thus we see that the derivative depends upon two terms:

$$\left( \frac{\bar{M}_{t_b} - \bar{M}_{t_a}}{\Delta t} \right)$$

which represents the rate of accumulation of  $x\text{-momentum}$  within the control volume and

$$\left( \frac{v_2 \delta m_{out} - v_1 \delta m_{in}}{\Delta t} \right)$$

which represents the net rate of outflow of  $x\text{-momentum}$  for the control region.

---

\*In the case of mass conservation we evaluated the time derivative of the mass,  $m$ , of the system. It is interesting to note further that that derivative equaled zero in accordance with the mass conservation law while the momentum derivative equals not zero but the force on the system in accordance with the 2nd Law of Motion.

Now

$$\delta m_{in} = (\rho_1 A_1 v_1) \Delta t$$

$$\delta m_{out} = (\rho_2 A_2 v_2) \Delta t$$

$$\text{so } dM/dt = \lim_{\Delta t \rightarrow 0} \left( \frac{\bar{M}_{t_b} - \bar{M}_{t_a}}{\Delta t} \right) + \left\{ \frac{(\rho_2 A_2 v_2^2 - \rho_1 A_1 v_1^2)}{\Delta t} \right\} \Delta t$$

and finally, remembering that the force on the mass equals this derivative, we get the "control volume expression of the 2nd Law of Motion" commonly called the momentum equation.

$$\boxed{(\text{Force on Mass})_x = \frac{d\bar{M}}{dt} + \rho_2 A_2 v_2^2 - \rho_1 A_1 v_1^2}$$

This is an expression for the force on the mass coincident with the control region at the instant  $t_a$ , since in the limit  $t_b = t_a$ .

If the flow through the control region is steady then there is no accumulation or diminution of momentum in the control volume and

$$\frac{d\bar{M}}{dt} = 0$$

So the momentum equation for steady one dimensional flow becomes

$$\boxed{\text{Force on Mass} = \rho_2 A_2 v_2^2 - \rho_1 A_1 v_1^2}$$

Concerning this equation of momentum Prandtl and Tietjens\* make the remark. "The undoubted value of the theorem of momentum lies in the fact that its application enables one to obtain results in physical problems from just a knowledge of the boundary conditions. There is no need to be told anything about the interior of the fluid or about the mechanism of the motion." This statement applies equally well to each of the "control volume equations."

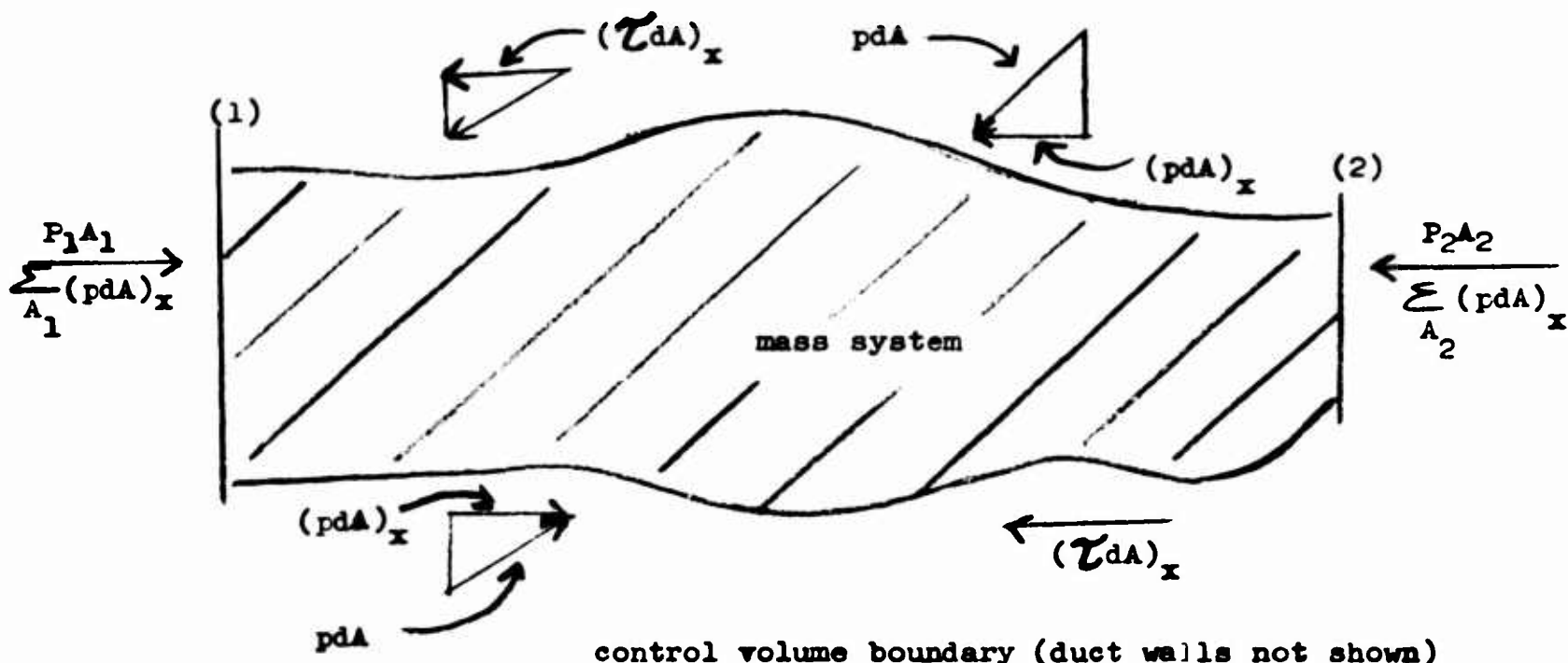
#### Application of Momentum Equation

Usually the situation is such that one is more interested in the force of fluid on duct between section (1) and (2) than in the force on the mass system. To obtain the former, denoted by script F, we observe that any force acting on the mass in the control region (neglecting gravity) will act at the control boundary and will be either a shearing force tangent to the boundary or a pressure force acting normal to the boundary. If then we make a traverse of the control boundary at a given instant we find the

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\*Prandtl-Tietjens, Fundamentals of Hydro-and Aero-Mechanics, page 233, McGraw-Hill, 1934.

forces depicted in the figure herewith to be acting on the mass system at that instant. Summing the



x-component of these forces over the control volume (c.v.) boundary we obtain

$$(\text{Force on mass})_x = \sum_{\text{c.v.}} (pdA)_x + \sum_{\text{c.v.}} (\tau dA)_x$$

where  $\tau$  is shearing stress at mass system boundary. Now, expanding the two summations

$$\sum_{\text{c.v.}} (pdA)_x = \sum_{\text{duct walls}} (pdA)_x + \sum_{A_1} (pdA)_x + \sum_{A_2} (pdA)_x$$

and

$$\sum_{\text{c.v.}} (\tau dA)_x = \sum_{\text{duct walls}} (\tau dA)_x$$

Therefore, substituting the expanded summations,

$$(\text{Force on mass})_x = \sum_{\text{duct walls}} (pdA)_x + \sum_{\text{duct walls}} (\tau dA)_x + \sum_{A_1} (pdA)_x + \sum_{A_2} (pdA)_x$$

Where the first two summations represent the force of duct on fluid ( $= -\mathcal{F}_x$ ). Using this fact then, the total (force on mass) $_x$  is seen to be made up of the following three forces

$$(\text{Force on mass})_x = -\mathcal{F}_x + p_1 A_1 - p_2 A_2.$$

Thus we have found two expressions for the (force on mass) $_x$ :

(a) the expression  $(\text{Force on mass})_x = -\mathcal{F}_x + p_1 A_1 - p_2 A_2$

which actually represents an identity obtained by examining the possible forces acting at the c.v. boundary and summing these forces and

(b) the equation  $(\text{Force on mass})_x = \rho_2 A_2 v_2^2 - \rho_1 A_1 v_1^2$

which was obtained by application of Newton's law of motion.

Combining (a) and (b) we obtain an equation for the force of fluid on duct ( $\mathcal{F}$ ).

$$\mathcal{F}_x = (p_1 A_1 + \rho_1 A_1 v_1^2) - (p_2 A_2 + \rho_2 A_2 v_2^2)$$

or

$$\mathcal{F}_x = F_1 - F_2$$

where  $F = pA + \rho A v^2$  and is called the impulse function. Notice that  $F$  is a function of the stream properties and area at any given section and is therefore a function of position along the stream.



Lecture Note 4 FIRST LAW OF THERMODYNAMICS

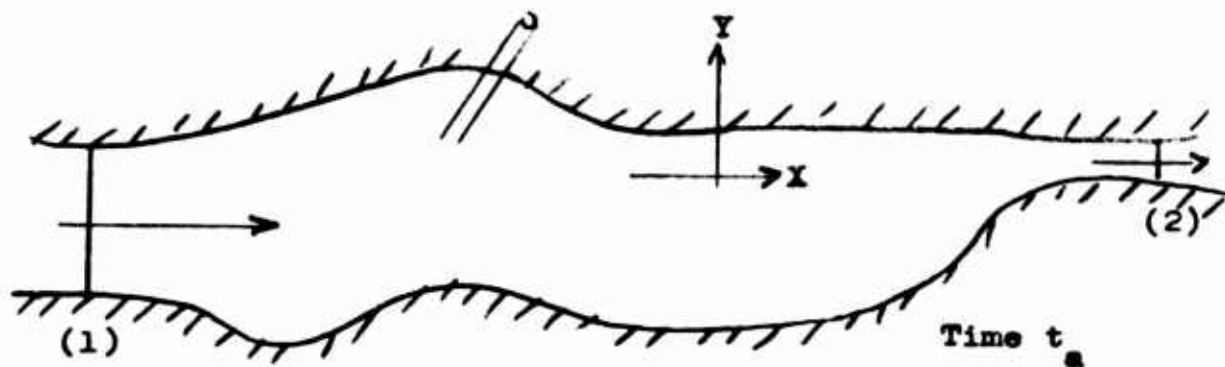
The law of conservation of mass and the 2nd Law of Motion have, in the preceding notes, been written for a fixed mass system following which the continuity and momentum equation were developed for a control volume. It is proposed in this note to follow the same procedure, or method of attack, in handling the 1st Law of Thermodynamics.

The first law of thermodynamics states, symbolically, for a mass of fixed identity

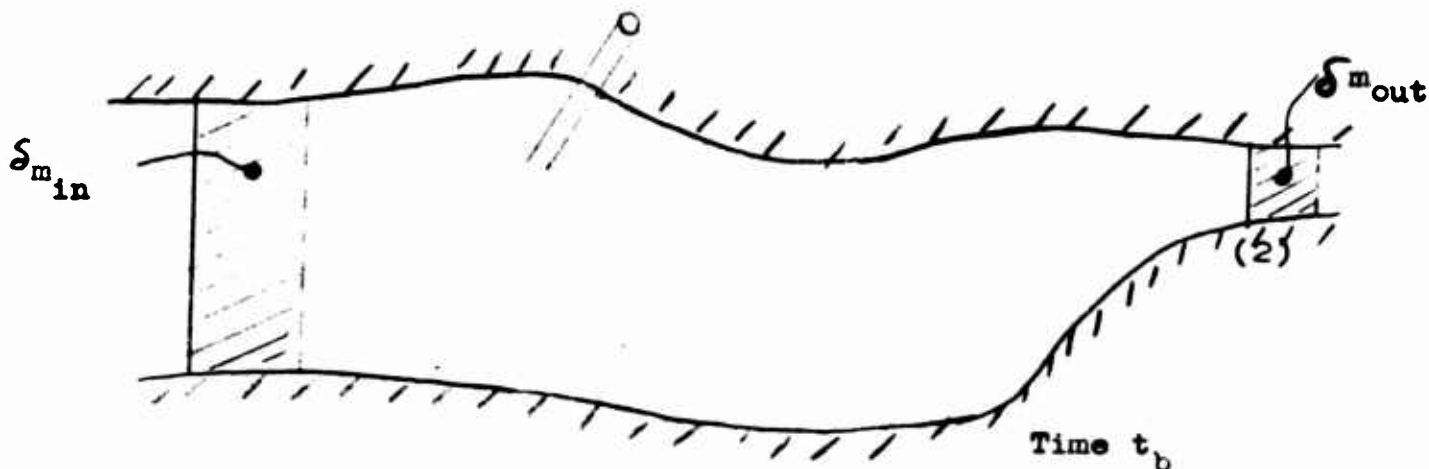
$$\text{Heat} - \text{Work} = E_b - E_a$$

where  $E_b - E_a$  is the change of internal energy of the mass system in state b and state a and where heat and work are, respectively, the amount of heat added to and the amount of work done by the system as it changes from state a to state b. Let us now use this statement to develop an equation applicable to fluid flow through a control volume.

Consider the control volume below bounded by the solid boundary walls and sections (1) and (2). In applying the first law let us select as our mass system that matter bounded by the control volume at time  $t_a$ . At time  $t_b$  this system has moved to the position shown. The change of internal energy of the system



Mass system contained within control volume



Mass system not coincident with control volume

during this change of state is (following the procedure used for mass and momentum)

$$E_{t_b} - E_{t_a} = \bar{E}_{t_b} - \bar{E}_{t_a} + \delta E_{out} - \delta E_{in}$$

where the bar symbol refers, as before, to the energy of the mass in the control volume. In order to evaluate  $\delta E_{out}$  and  $\delta E_{in}$  we simply multiply the mass increments that have flowed out and into the control volume in time  $\Delta t$  by the internal energy per unit mass of their respective increments. Thus

$$\delta E_{out} - \delta E_{in} = (\rho_2 A_2 V_2 \Delta t) e_2 - (\rho_1 A_1 V_1 \Delta t) e_1$$

so we have by substituting into equation (1)

$$\frac{\text{Heat-Work}}{\Delta t} = \frac{\bar{E}_{t_b} - \bar{E}_{t_a}}{\Delta t} + (\rho_2 A_2 V_2) e_2 - (\rho_1 A_1 V_1) e_1$$

and for steady flow, with  $w = \rho_2 A_2 V_2 = \rho_1 A_1 V_1$  and  $\bar{E}_{t_b} = \bar{E}_{t_a}$ .

$$\frac{\text{Heat-Work}}{w \Delta t} = \frac{\text{Heat-Work}}{\text{unit mass}} = e_2 - e_1$$

We know by experience that the energy associated with a unit of mass in the presence of a gravitational field and motion is

$$e = u + \frac{V^2}{2} + Z g$$

where  $u$  is the internal energy of a unit mass in the absence of potential-kinetic effects and  $g$  is the acceleration of gravity. Thus we have

$$\frac{\text{Heat-work}}{w \Delta t} = \left( u_2 + \frac{V_2^2}{2} + Z_2 g \right) - \left( u_1 + \frac{V_1^2}{2} + Z_1 g \right) \quad (2)$$

Up to this point we have considered only the right hand side of equation (1). Let us examine next the left side and in particular the work term. As the mass system passes from state (a) to state (b) work is done on the system boundaries (which move to the dashed positions of state b) by pressure forces. At the same time there may be work done by the system through a shaft protruding through the control surface. Thus we may write for the work term

$$\text{Work} = \text{pressure force work} + \text{shaft work}$$

The pressure force work at section (1) is

$$\begin{aligned} & (\text{pressure} \times \text{area})_1 \times (\text{distance moved})_1 \\ & = p \Delta s = -p \left( \frac{A_1 \Delta s_1}{\delta m_1} \right) \delta m_1 = -p_1 v_1 \delta m_1 \end{aligned}$$

where  $v_1 = \text{specific volume}$   $\left[ \frac{\text{ft}^3}{\text{slug}} \right]$ .

Since work is done on the system by the pressure force at (1) a minus sign is included above. In like manner we find the pressure force work at (2). Whence

$$\text{pressure work} = p_2 v_2 \delta m_2 = p_1 v_1 \delta m_1.$$

Since  $\delta m = \text{density} \times \text{volume} = \rho A V \Delta t = w \Delta t$  we may write

$$\frac{\text{pressure work}}{w \Delta t} = p_2 v_2 - p_1 v_1$$

Now the left hand side of equation (2) takes the form

$$\frac{\text{Heat-Work}}{w \Delta t} = \frac{\text{Heat-pressure work-shaft work}}{w \Delta t} = \frac{\text{Heat-Shaft work}}{w \Delta t} - (p_2 v_2 - p_1 v_1)$$

This result combined with the right hand side of equation (2) gives after transposing and using the definition of enthalpy,  $h = u + pv$ ,

$$q - w_x = \left( h_2 + \frac{v_2^2}{2} + z_2 g \right) - \left( h_1 + \frac{v_1^2}{2} + z_1 g \right)$$

where  $q = \text{heat transfer per unit mass}$

$w_x = \text{shaft work per unit mass.}$

This equation is called the steady flow energy equation and is the mathematical form of the 1st Law which applied to flow through a control volume. It may be of interest to note at this point that two fundamental laws are used to obtain the steady flow energy equation - the 1st Law of Thermodynamics and Newton's 2nd Law of Motion. The latter enters in the development of the kinetic energy term  $v^2/2$  which, of course, was not covered in this note.

## Lecture Note 5

COMBINATION OF THE LAWS OF THERMODYNAMICS AND OF FLUID MECHANICS  
FOR INCOMPRESSIBLE FLUID FLOW

For a steady flow of a single fluid stream through a control surface fixed in space, the first law of thermodynamics and Newton's second law of motion yield the energy equation for steady flow:

$$(u_2 + p_2 v_2 + \frac{v_2^2}{2} + gz_2) - (u_1 + p_1 v_1 + \frac{v_1^2}{2} + gz_1) = Q - W_x \quad (1)$$

where  $Q$  denotes the heat transfer into the control volume per unit mass of flowing fluid,  $W_x$  denotes the shaft work delivered out of the control volume per unit mass of flowing fluid, the subscript 2 refers to the stream leaving the control surface, and the subscript 1 refers to the stream entering the control surface.

If the sections 1 and 2 are so close to each other that only infinitesimal effects occur, we may write the equation in differential form:

$$du + d(pv) + d\frac{v^2}{2} + gdz = dQ - dW_x \quad (2)$$

or, since

$$d(pv) = pdv + vdp \quad (3)$$

equation (2) may be written

$$du + pdv + vdp + d(v^2/2) + gdz = dQ - dW_x \quad (4)$$

or, transposing some terms,

$$dW_x + vdp + d(v^2/2) + gdz = dQ - du - pdv \quad (4a)$$

For a pure substance we have the following relation between properties, where  $s$  is the entropy:

$$Tds = du + pdv \quad (5)$$

The second law of thermodynamics may be introduced by the relation

$$dQ \geq Tds \geq du + pdv \quad (6)$$

which, when inserted into equation (5), yields

$$dQ - du - pdv \geq 0 \quad (7)$$

Combining equations (7) and (4a), we obtain

$$dW_x \leq - [vdp + d(v^2/2) + gdz]$$

or, in integral form,

$$W_x \leq - \left[ \int_1^2 vdp + \frac{v_2^2 - v_1^2}{2} + g(z_2 - z_1) \right] \quad (8)$$

This equation, which places a limit on the maximum shaft work which may be delivered, is not easily evaluated in practice because one seldom knows how the density varies with pressure. If the fluid is incompressible, however, we obtain immediately (using  $\rho = 1/v$ ):

$$W_x = \left( \frac{p_1}{\rho} + \frac{v_1^2}{2} + gz_1 \right) - \left( \frac{p_2}{\rho} + \frac{v_2^2}{2} + gz_2 \right) \quad (9)$$

or, introducing the definition of the head,  $H$ , in length units,

$$H = \frac{p}{\rho g} + \frac{v^2}{2g} + z \quad (10)$$

we have

$$W_x \leq g(H_1 - H_2) \quad (11)$$

Under conditions of thermodynamically reversible flow, for which friction would be excluded, only the equality sign is applicable in equation (6), and hence equation (11) becomes

$$(W_x)_{\text{rev}} = g(H_1 - H_2) \quad (12)$$

Furthermore, if the shaft work is zero between sections 1 and 2, equation (12) shows that for reversible flow the head  $H$  is constant. This result is essentially the statement of the Bernoulli equation, since the head  $H$  as defined by equation (10) is identical with what is called the Bernoulli number of the streamline.

The form of equation (11) suggests that we define the "lost head,"  $H_{l12}$  by the expression

$$H_{l12} = \frac{1}{g}(W_x)_{\text{rev}} - W_x = H_1 - H_2 - \frac{W_x}{g} \quad (13)$$

By comparison of equations (11) and (13) it is evident that  $H_{l12}$  must always be a positive number or zero. The lost head is associated with frictional effects, and its magnitude may usually be found only through experiment.

Application to Flow in Piping Systems. A piping system usually comprises straight lengths of pipe, elbows, reducers, and other fittings. There are losses of head through these various components, and to keep the fluid flowing requires the use of a pump, compressor, or fan. An important design problem therefore, is to estimate the total loss in head through the piping system.

Since there is no shaft work associated with any element of a piping system equation (13) becomes

$$H_1 - H_2 = H_{l12} \quad (14)$$

For the fluid velocities commonly used in engineering practice, the lost head in a component is approximately proportional to the velocity head, hence we may define

$$C = \frac{d}{v^2/2g} H_{l12} \quad (15)$$

where  $C$  is a lost-head coefficient. The value of  $C$  depends to some extent on the velocity, density, nature and size of the fitting, and nature of the approach flow to the fitting, but in most cases  $C$  changes by only small amounts with changes in these variables.

Approximate values for  $C$  for various types of fittings are listed below:

<u>Fitting</u>	<u>C</u>
45-deg. elbows	0.3-0.4
90-deg. elbows, standard radius	0.7-0.9
90-deg. elbows, medium radius	0.5-0.8
90-deg. elbows, long sweep	0.4-0.6
90-deg. square elbows	1.0-2.0
Tee	1.0-2.0
180-deg. return bend	1.0-2.0
Open gate valve	0.1-0.2
Open globe valve	6-9
Open angle valve	3-5
Sudden contraction from infinitely large pipe	0.5
Sudden expansion to infinitely large pipe	1
Straight pipe	$0.01 \frac{L}{D} - 0.04 \frac{L}{D}$

For the straight pipe,  $L$  refers to the length of pipe and  $D$  to the diameter.

The total loss in head for a complete piping system is the sum of the losses for the individual components.

Application to Hydraulic Turbines The efficiency of a hydraulic turbine is defined as the ratio of the actual shaft work delivered to the work which would be delivered under reversible conditions for the same change in head:

$$\eta_{\text{turb}} = \frac{W_{\text{turb}}}{(W_{\text{turb}})_{\text{rev}}} \quad (16)$$

From equation (12), however,

$$(W_{\text{turb}})_{\text{rev}} = g(H_1 - H_2)$$

so that

$$\eta_{\text{turb}} = \frac{W_{\text{turb}}}{g(H_1 - H_2)} \quad (17)$$

Using equation (13), the turbine efficiency may be expressed in terms of the losses in the turbine,

$$\eta_{\text{turb}} = \frac{W_{\text{turb}}}{g(H_1 - H_2)} = \frac{W_{\text{turb}}}{W_{\text{turb}} + gH_{\text{turb}}} = \frac{1}{1 + \frac{gH_{\text{turb}}}{W_{\text{turb}}}} \quad (18)$$



from which we get

$$\frac{gH}{w_{\text{turb}}} = \frac{1 - \eta_{\text{turb}}}{\eta_{\text{turb}}} \quad (19)$$

Application to Pumps and Fans. For incompressible flow, the efficiency of a pump or fan is defined as the ratio of the reversible shaft work input for a given increase in head to the actual shaft work input to the machine:

$$\eta_{\text{pump}} = \frac{(w_{\text{pump}})_{\text{rev}}}{w_{\text{pump}}} \quad (20)$$

from equation (12), however, we may write, upon noting that the thermodynamic shaft work is the negative of the work input, that

$$(w_{\text{pump}})_{\text{rev}} = g(H_2 - H_1)$$

so that

$$\eta_{\text{pump}} = \frac{g(H_2 - H_1)}{w_{\text{pump}}} \quad (21)$$

We now relate the efficiency to the losses by introducing equation (13) in the form

$$w_{\text{pump}} = -w_x = g(H_2 - H_1 + h_{\text{pump}})$$

so that we obtain

$$\eta_{\text{pump}} = \frac{w_{\text{pump}} - gH_{\text{pump}}}{w_{\text{pump}}} = 1 - \frac{gH_{\text{pump}}}{w_{\text{pump}}} \quad (22)$$

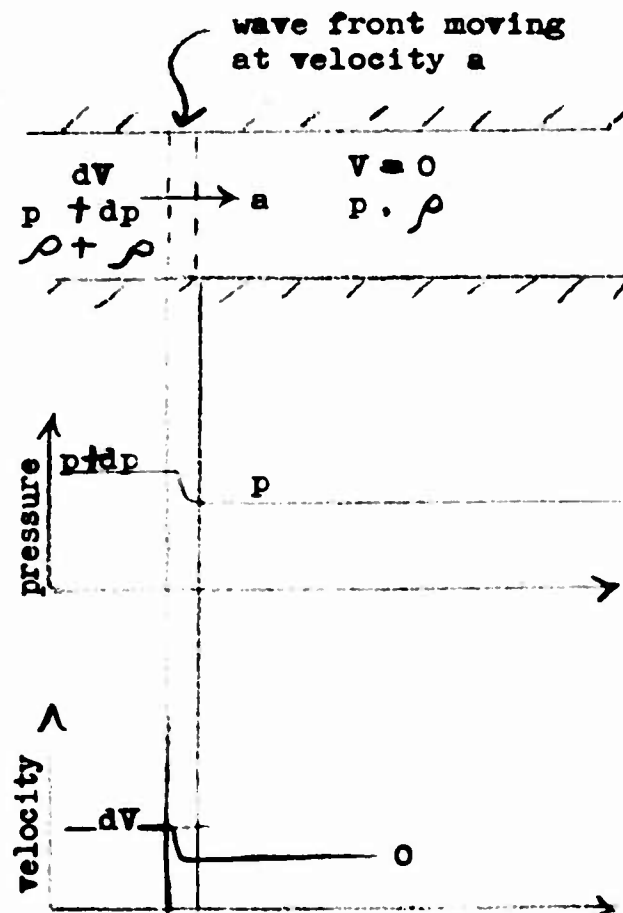
from which we get

$$\frac{gH_{\text{pump}}}{w_{\text{pump}}} = 1 - \eta_{\text{pump}} \quad (23)$$

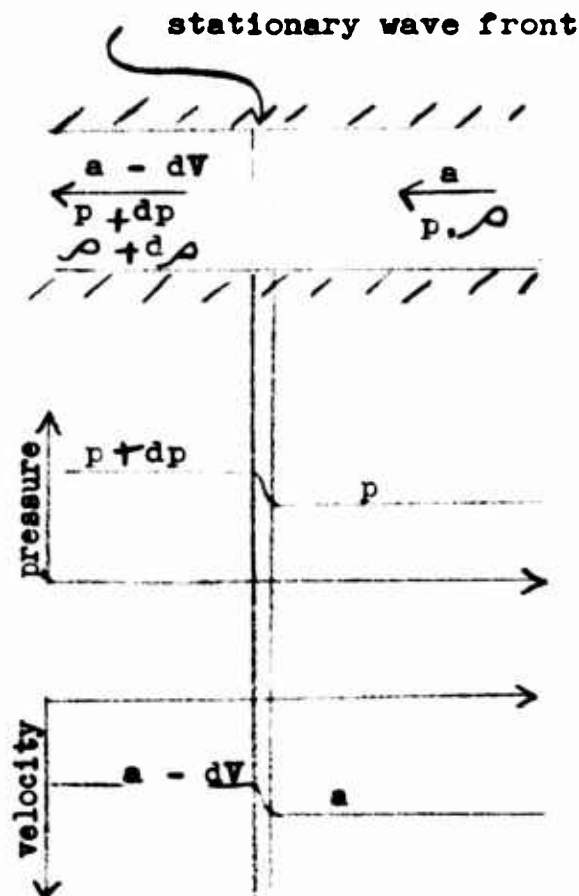
Lecture Note 6 THE VELOCITY OF SOUND (1)

The variation of fluid density in a compressible fluid flow field is generally the result of pressure variation throughout the flow field. It might be expected therefore that the rate of change of density with respect to pressure ( $d\rho/dp$ ), compressibility factor, is an important parameter in compressible fluid flow studies. Such is the case and, as we shall see, this derivative is connected closely with the propagation velocity of small disturbances its reciprocal being equal, in fact, to the square of velocity of sound ( $dp/d\rho = a^2$ ).

Let us now develop the speed of sound or the velocity of an infinitesimal pressure wave proceeding along a pipe of uniform cross-section. This wave might be considered to have been initiated, for example, by a slight inward motion of a piston at the left hand end of the pipe. The development to follow will illustrate also the application of the "control volume equations"



Variation of pressure and velocity of fluid at a given instant of time. (Observe at rest with respect to duct)

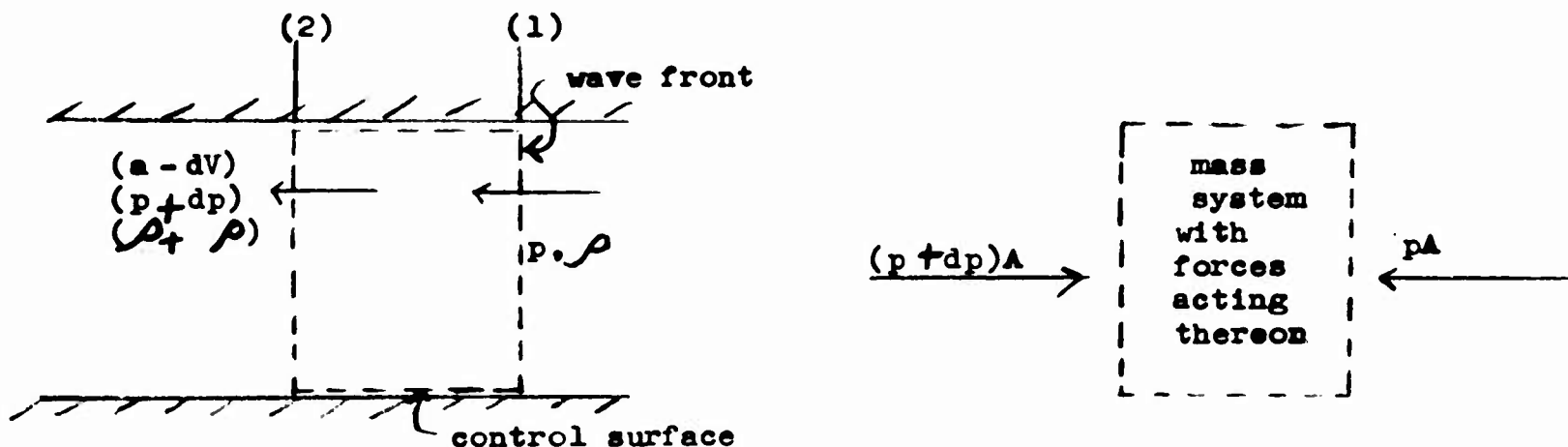


Variation of pressure and velocity of fluid at any instant (Observer at rest with respect to wave front)

In the figure on the left above the wave front is assumed to propagate to the right with a velocity  $a$ . The fluid through which the wave front has passed is at a pressure  $(p + dp)$ , has a density  $(\rho + d\rho)$ , and moves to the right with a velocity  $dV$ . The fluid into which the wave is propagating has a pressure  $p$ , a density  $\rho$ , and is motionless. This frame of reference is one of "unsteady motion" since as time progresses the stream properties at a given dust section vary with time.

To simplify the analysis let us assume the point of view of one traveling with the wave. To this observer the wave appears at rest and the process appears steady as shown on the right in the figure above. Fluid flows steadily from right to left approaching the wave front at a velocity  $a$  and leaving with a velocity  $(a - dV)$  while the fluid pressure and density changes from  $p$  and  $\rho$  to  $p + dp$  and  $\rho + d\rho$ , respectively, across the wave.

For purposes of analysis consider the infinitesimal wave front many times enlarged and draw a control surface about the wave front region to get control volume shown in the figure below.



Let us apply the momentum and continuity equation to the steady flow through this control volume. We have by the momentum equation (taking direction of fluid velocity to left as positive and denoting the inlet section as 1 with outlet indicated by subscript 2)

$$(\text{Force on mass})_x = \rho_1 A_1 V_1 (V_2 - V_1) \quad (1)$$

While the continuity equation states

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 \quad (2)$$

For the case under consideration

$$\rho_1 = \rho$$

$$V_1 = a$$

$$\rho_2 = \rho + d\rho$$

$$V_2 = a - dV$$

$$A_1 = A_2 = A$$

$$(\text{Force on mass})_x = pA - (p + dp)A$$

Substituting these into equation (1) and (2) gives

$$pA - (p + dp) A = \rho A (a - dV - a) \quad (1a)$$

$$\rho A a = (\rho + d\rho) A (a - dV) \quad (2a)$$

Simplifying

$$dp = \rho a dV \quad (1b)$$

$$0 = -\rho dV + a d\rho - (d\rho)(dV) \quad \text{H.O.T.} \quad (2b)$$

or, finally,

$$\frac{dp}{d\rho} = a^2; \quad a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}$$

The ratio  $\frac{dp}{d\rho}$  is written as a partial derivative at constant entropy because the variation in pressure and temperature are very small and, consequently, the process is nearly reversible. Moreover, the rapidity of the process, together with the smallness of the temperature variations, makes the process nearly adiabatic. In the limit the process may be considered both reversible and adiabatic, and therefore, isentropic.

For a perfect gas we have the isentropic relation

$$\left(\frac{p}{\rho}\right)^k = \text{constant}.$$

Putting this into logarithmic form, differentiating, and noting that  $p = \rho RT$ , we obtain

$$\ln p - k \ln \rho = \text{constant}$$

$$\frac{dp}{p} = k \frac{d\rho}{\rho}; \quad \left(\frac{\partial p}{\partial \rho}\right)_s = k \frac{p}{\rho} = kRT$$

Thus we get for the velocity of sound in a perfect gas

$$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} = \sqrt{kRT}.$$

In the case of air with  $k=1.4$   $R=1715 \frac{\text{ft}^2}{\text{sec}^2 \text{ } ^\circ\text{R}}$  this becomes

$$a = 49.1 \sqrt{T} \quad (T \text{ in } ^\circ\text{R})$$

where units to be associated with 49.1 are  $\frac{\text{ft}}{\text{sec}} \sqrt{^\circ\text{R}}$

### Pressure Propagation from a Point Disturbance

The physical significance of the sound velocity may be illustrated by considering the uniform linear motion of a point source of disturbance through a compressible medium. At each instant of time the point source may be imagined to emit an infinitesimal pressure wave which spreads spherically with the speed of sound from the point of emission. The pressure pattern which exists at any instant is then found by superposition of all the pressure pulses which were previously emitted.

The accompanying figure shows several patterns as seen by an observer moving with the point disturbances. In each pattern the point 0 represents the present location of the point disturbance, the point -1 represents the location one unit of time previously and so on. For each of these previous locations there is drawn a concentric circle showing the extents to which the corresponding wave has spread. For example, to find the present location of the wave which was emitted at time -3 a circle is drawn with -3 as a center and with a radius  $3at$ , where  $t$  is the unit of time. The distance between point -3 and point 0 is then given by  $3Vt$ , where  $V$  is the velocity of the point disturbance with respect to the medium.

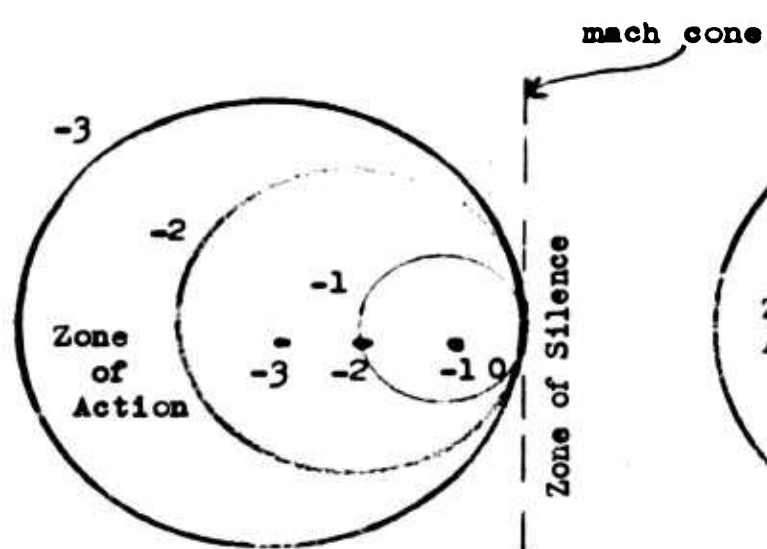
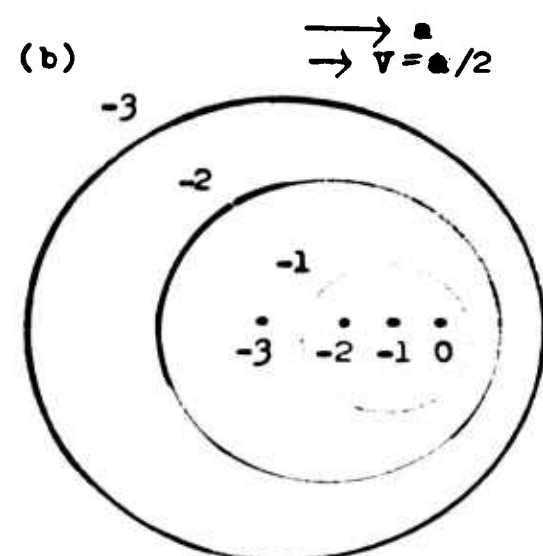
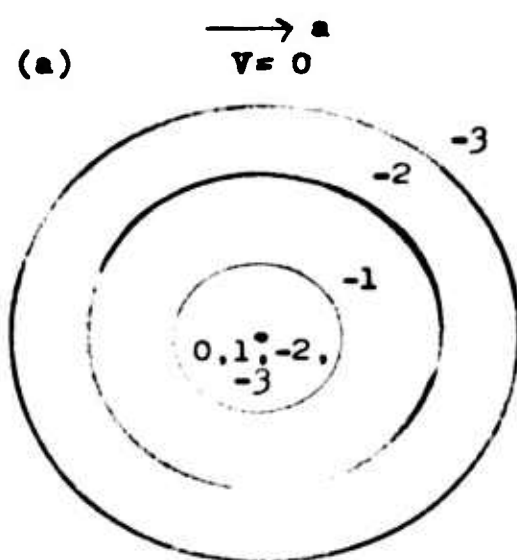
For a stationary source, shown in Figure (a), the pressure change spreads uniformly in all directions. When the source moves at subsonic speeds, Figure (b), the pressure disturbance is felt in all directions and at all points in space (neglecting dissipation due to viscosity) but the pressure pattern is no longer symmetrical.

For supersonic speeds Figure (d) indicates that the phenomena are entirely different from those at subsonic speeds. All the pressure disturbances are included in a cone which has the point source as its apex, and the effect of the disturbance is not felt upstream of the source of disturbance. The cone within which the disturbances are confined is called the Mach cone. Figure (c) shows the pressure pattern at the boundary between subsonic and supersonic flow, that is, for the case where the stream velocity is identical with the sound velocity.

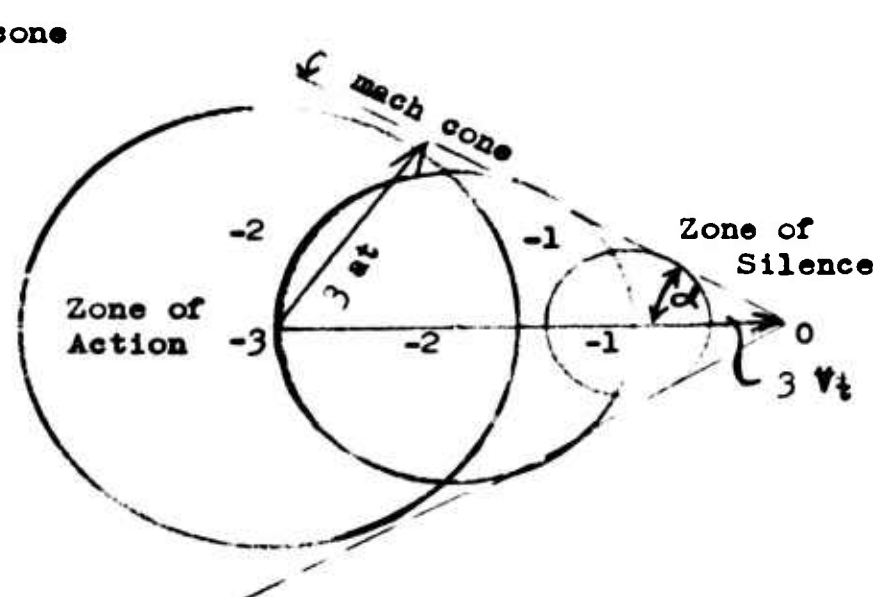
Figure (d) illustrates the three rules of supersonic flow proposed by vonKarman\*. These rules apply only for small disturbances, but are usually qualitatively applicable for large disturbances.

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\*"Supersonic Aerodynamics--Principles and Application", by Th. vonKarman, Journal Aero. Sci., Vol. 14, no. 7 (1947) pg. 373



(c)  $\vec{v} = a$



(d)  $\vec{v} = 2a$



a. The Rule of Forbidden Signals. The effect of pressure changes produced by a body moving at a speed faster than sound cannot reach points ahead of the body.

b. The Zone of Action and the Zone of Silence. A stationary point source in a supersonic stream produces effects only on points that lie on or inside the Mach cone extending downstream from the point source. Conversely, the pressure and velocity at an arbitrary point of the stream can be influenced only by disturbances acting at points that lie on or inside a cone of the same vertex angle extending upstream from the point considered.

c. The Rule of Concentrated Action. The proximity of the circles representing the different pressure impulses in the figure is a measure of the intensity of the pressure disturbance at each point in the field of flow. Thus, for the stationary source, the intensity of the disturbance is symmetrical. In the case of the supersonic source, we have the rule of concentrated action: the pressure disturbance is largely concentrated in the neighborhood of the Mach cone that forms the outer limit of the zone of action.

The configurations shown may easily be observed in the form of gravity waves on a free water surface when a sharp-pointed object is drawn through the water at varying speeds.

### The Mach Number

In the preceding section it was shown that the nature of the flow pattern depends on the relation between the stream velocity and the sound velocity. The ratio of these two velocities is called the Mach Number. Thus,

$$M = V/a$$

The speed of sound in this equation is to be taken at the local temperature and pressure of the stream, and, of course, varies from point to point in the flow field.

The semi-angle of the Mach cone (figure d) is related to the Mach Number as follows:

$$\sin \alpha = \frac{1}{M}$$

Note that the Mach angle is imaginary for subsonic flow.

From the preceding section we see that the Mach Number is a criterion of the type of flow pattern. Later it will be shown that it is a convenient parameter that will appear in our working equations.

## Lecture Note 7

## TOTAL PRESSURE AND TOTAL TEMPERATURE\*

The purpose of this note is to introduce the concept of total temperature,  $T_o$ , and total pressure,  $p_o$ , and to show that the ratios of static to total temperature ( $T/T_o$ ) and static to total pressure ( $p/p_o$ ) are each functions of Mach Number.

Total Temperature

Consider the steady flow energy equation

$$\left[ h_1 + \frac{v_1^2}{2} \right] + Q = \left[ h_2 + \frac{v_2^2}{2} \right] + W_x.$$

The kinetic energy terms may be combined with enthalpy to form a new term, total enthalpy,  $h_o$ . Thus

$$h_o = h + \frac{v^2}{2}.$$

If the flow under consideration is that of a perfect gas, then

$$d h = c_p dT$$

$$\text{and } d h_o = c_p dT + d \frac{v^2}{2}$$

$$= c_p d \left( T + \frac{v^2}{2c_p} \right)$$

$$\text{or } d h_o = c_p dT_o$$

Where  $T_o = T + \frac{v^2}{2c_p}$  and is defined as the total temperature.

The physical significance of total temperature may be illustrated by the use of the following figure. If in the figure an observer should travel with the slug of gas shown at the same velocity as the gas he would be cognizant only of the random motion of the molecules. Hence, since the static temperature and pressure result from the random motion of the gas molecules, the observer would sense static values of temperature and pressure. In a flowing gas the molecules have superimposed on their random motion the directed motion of the flow. The kinetic energy of the directed motion is the cause of the difference between the static and total temperature.

$$T_o - T = \frac{v^2}{2c_p}$$

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\* Reference: pp. 20-21, AAF TR 5514

## NOZZLE DESIGN

By simple area flow we mean the one dimensional flow of a perfect gas in the absence of friction or heating effects. This type of flow satisfies the following conditions:

1. frictionless
2. adiabatic
3. one dimensional

A simple area type of flow may be used to accelerate the stream flow or to decelerate the flow velocity. The flow passages producing these effects are called nozzles (accelerate flow) and diffusers (decelerate flow) respectively.

Consider a simple area flow. The equations satisfied by this flow are

$$p = \rho RT \quad (\text{state})$$

$$T_1 + \frac{V_1^2}{2c_p} = T_2 + \frac{V_2^2}{2c_p} = T_0 = \text{constant} \quad (\text{energy})$$

$$\dot{m} = \rho AV \quad (\text{continuity})$$

$$\frac{p}{p_1} = \left( \frac{T}{T_1} \right)^{\frac{k}{k-1}} \quad (\text{2nd Law})$$

where subscript (1) designates conditions at the inlet to the flow being considered and no subscript denotes any station downstream from the inlet. Since there are five variables in the above four equations ( $p$ ,  $\rho$ ,  $T$ ,  $V$ , and  $A$ ) we may select one as an independent variable and find each of the remaining four in terms of this one. Practical problems generally fall into either one of two classes.

- (a) It is desired to pass a given mass rate of flow with minimum losses between two regions of different pressures with some assumed variation of pressure, say linear, between the two regions.
- (b) Given a nozzle, what mass rate of flow and pressure distributions will exist through this passage of variable area for various pressure ratios applied across the unit?

In case (a) our independent or known variable is pressure,  $p$ . In case (b) our quantity of known variation is area,  $A$ . We shall consider each case in turn.

As an illustration of case (a) consider the following example.

Example (a) It is desired to expand 0.62 slugs of air per sec. reversibly and adiabatically between a reservoir and exhaust region with following conditions.

$$p_1 = 300 \text{ psia}$$

$$\text{Passage length} = 5''$$

$$T_1 = 560^\circ \text{ R}$$

$$\text{Exhaust region pressure} = 40 \text{ psia}$$

$$V_1 = 100 \text{ ft/sec.}$$

$$\text{Linear variation of pressure from reservoir to discharge region}$$

$$w = 0.62 \text{ slugs/sec}$$

Design a nozzle to meet above requirements.

Solutions: Of the five variables in the four applicable equations one, the pressure, is known throughout the flow. Hence we have four equations in four unknowns. To determine the area at any particular station we proceed as follows:

Combine continuity and state equations and evaluate  $p_0$  and  $T_0$  then

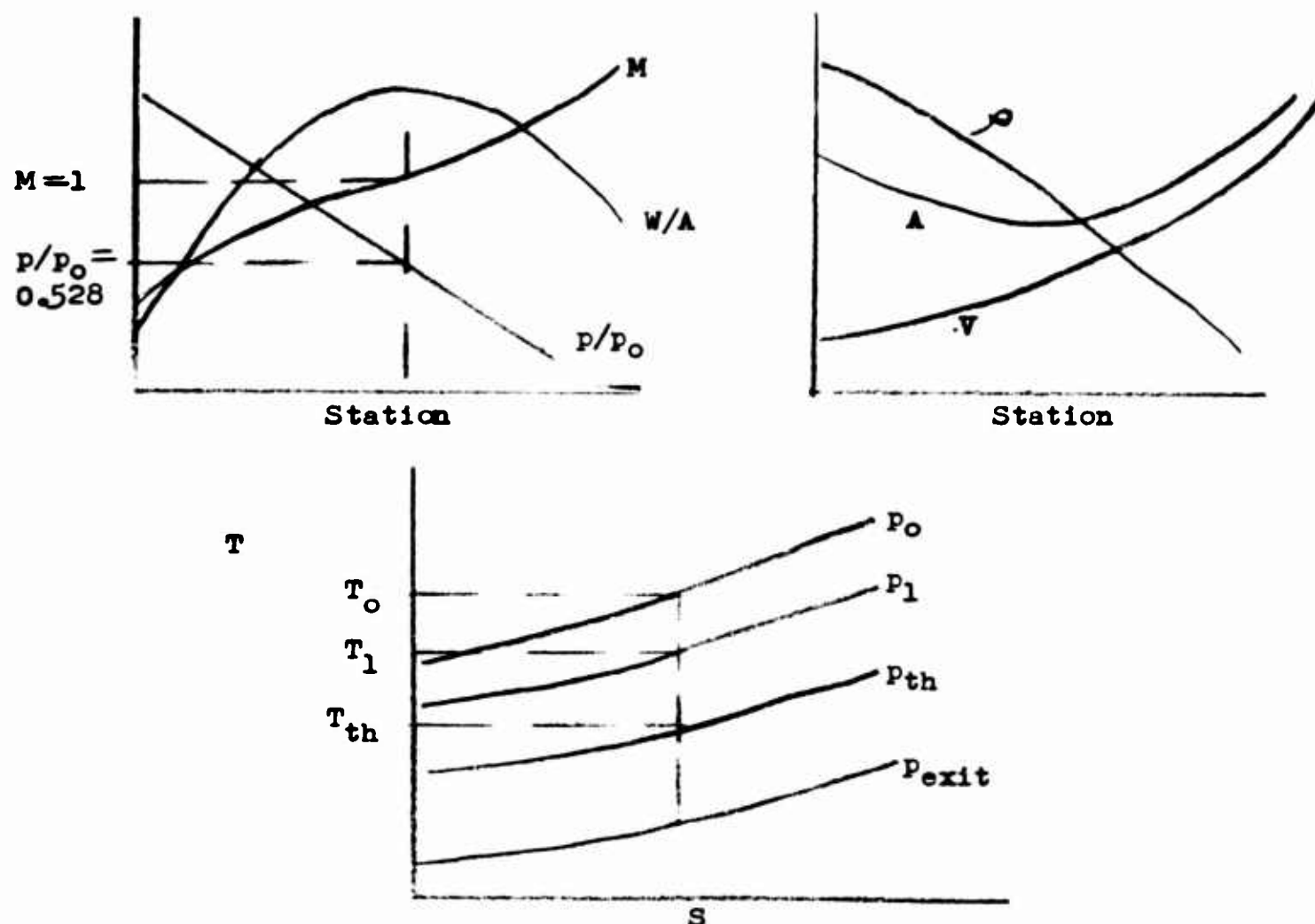
$$w = \frac{pAV}{RT} \quad (1) \quad (\text{state and continuity})$$

$$T_0 = T + \frac{V^2}{2c_p} \quad (2) \quad (\text{energy})$$

$$T = T_0 (p/p_0)^{\frac{k-1}{k}} \quad (3) \quad (\text{2nd Law})$$

With  $p$  known, use (3) to find  $T$  at any given station. Then equation (2) gives  $V$  at this station. For these values of  $T$  and  $V$  along with the known values of  $w$  and  $R$  equation (1) gives the requisite area of the nozzle at the selected station. And so forth for any station.

The results of the example may be summarized in the form of a plot  $p/p_0$ ,  $w/A$ ,  $A$ ,  $V$  and Mach number,  $M$ , versus nozzle station along with a  $t$ - $s$  diagram of the expansion process.



These graphs illustrate

- To decrease pressure, sections of decreasing area are required until a pressure of  $p = 0.528 p_0$  is reached. For reduction of the stream pressure below this value a passage of diverging area is required.
- For  $p/p_0 > 0.528$  we have  $M < 1$  and for  $p/p_0 < 0.528$  we have  $M > 1$  which indicates that in subsonic flow the pressure decreases with decreasing area and vice versa for supersonic flow.
- The stream velocity increases continuously through the nozzle. Thus we may say that in subsonic flow a converging area accelerates the flow and that a diverging area accelerates the flow in supersonic flow.
- The area decreases to a minimum (throat) and then increases.
- At the throat of the nozzle  $M = 1$ ,  $p/p_0 = 0.528$ , and, obviously,  $w/A$  is a maximum.

The expansion process through the nozzle is shown as a solid vertical line on the T-s diagram from the pressure  $p_1$  to the exit pressure. The value of the stream pressure and temperature at the throat of the nozzle are indicated on the diagram by  $p_{th}$  and  $T_{th}$ .

Having designed a nozzle to meet certain operating conditions, it is now of practical and academic interest to investigate the characteristics of the nozzle when operating at other than designed conditions, for example

$w \neq 0.62$  slugs/sec.,  $p_1 \neq 300$  psia, and/or  $p_{\text{exhaust region}} \neq 40$  psia.

This problem comes under case (b) noted above and will be considered next.



## Lecture Note 9

## NOZZLE OPERATING CHARACTERISTICS

As an illustration of the type of problem coming under class (b) as listed in the preceding lecture note consider home problem 9.1 which deals with the following. Given a nozzle with known inlet total pressure and total temperature at what mass rates of flow and for what corresponding exhaust region pressures will it operate reversibly and adiabatically?

Probably the simplest way to investigate this question is to deal with a single equation which in itself contains the restrictions placed on the flow by continuity, 1st Law, equation of state and 2nd Law. The four applicable equations may be combined into a single equation as follows. We have

$$w = \frac{pAV}{RT} \quad (1) \quad (\text{cont. and state})$$

$$T_0 = T + \frac{V^2}{2c_p} \quad (2) \quad (\text{energy})$$

$$p = p_0 \left( \frac{T}{T_0} \right)^{k/k-1} \quad (3) \quad (\text{2nd Law})$$

Equation (1) may be written

$$\frac{w}{A} = \frac{pV}{RT}$$

wherein

$$p = \left( \frac{p}{p_0} \right) p_0$$

$$V = \sqrt{2c_p (T_0 - T)} \quad \sqrt{2c_p T_0 \left[ 1 - \frac{p}{p_0} \right]^{k-1/k}}$$

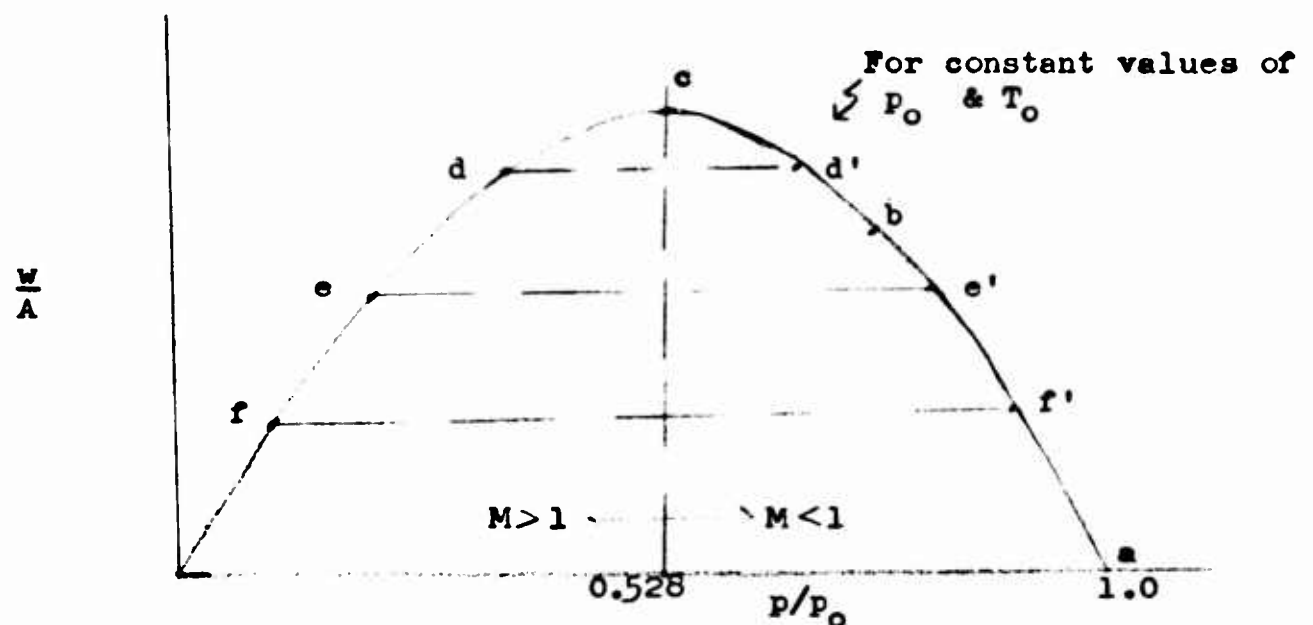
and

$$T = T_0 \left( \frac{p}{p_0} \right)^{k-1/k}$$

Substituting these expressions for p, V, and T we obtain after simplifying

$$\frac{w}{A} = \frac{p_0}{\sqrt{T_0}} \sqrt{\frac{2}{R} \frac{k}{k-1} \left[ \left( \frac{p}{p_0} \right)^{2/k} - \left( \frac{p}{p_0} \right)^{\frac{k+1}{k}} \right]}$$

If then this equation is satisfied at every section of the flow, it follows that the conditions imposed upon the flow by the 1st Law, 2nd Law, continuity equation, and state equation are satisfied. With  $p_0$  and  $T_0$  known in any given flow we may effect a graphical solution of the above equation by plotting  $(w/A)$  versus  $(p/p_0)$  where the latter may in the physical problem vary from 0 to 1. A graph of the relation  $w/A = f(p/p_0)$  is given below for, of course, some assumed value of  $p_0$  and  $T_0$ .



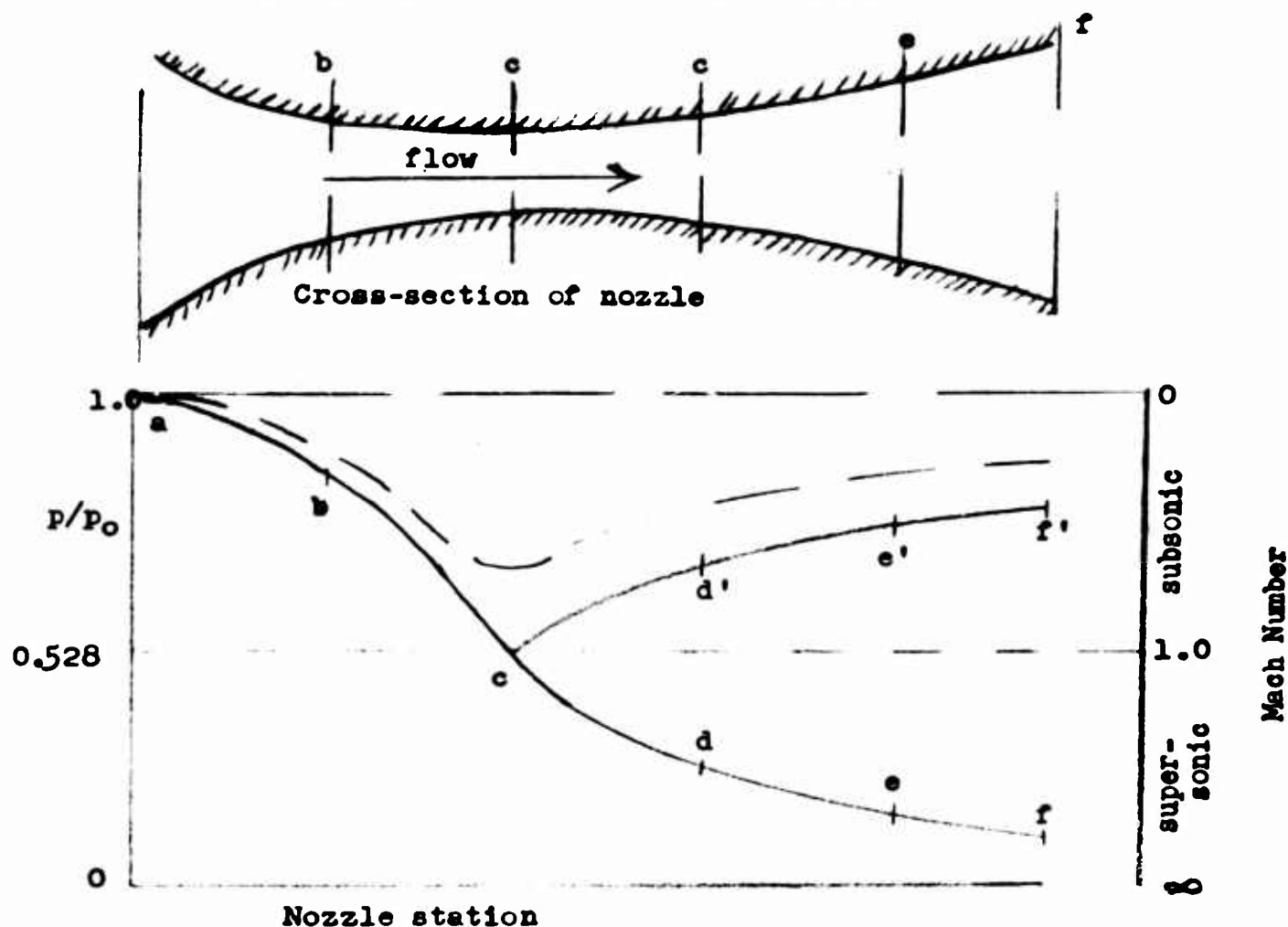
With the above plot values of  $w/A$  and  $p/p_0$  satisfying the equation  $w/A = f(p/p_0)$  may be easily found. The ratio of  $w/A$  may be determined at any station of a given nozzle with  $w$  known. Entering the graph with this predetermined value of  $w/A$  we find the value of  $(p/p_0)$  that must exist at the nozzle station selected. The ratio of  $p/p_0$  along with  $T_0$  and  $p_0$  fix the state of the fluid at this section.

It is to be noticed on the plot that, for a given value of  $(w/A)$ ,  $(p/p_0)$  is not uniquely determined. In any particular problem we can however by examining the physical aspects of the flow, determine which value of  $(p/p_0)$  is applicable.

As an illustration assume a nozzle is discharging air from a large reservoir isentropically with maximum mass rate of flow existing through nozzle. Plot the pressure distribution through this nozzle.

With the reservoir pressure and temperature known a plot of  $w/A$  versus  $p/p_0$  may be made. Then with mass rate of flow through nozzle known we can for nozzle sections b, c, d, e, and f measure area and determine  $(w/A)$ . With this value of  $w/A$  corresponding values of  $p/p_0$  are read from  $w/A - p/p_0$  plot above.

Beginning at the reservoir  $p/p_0 = 1$ , then as  $w/A$  increases  $p/p_0$  decreases from a to b to c at throat as indicated on  $w/A - p/p_0$  plot. After reaching the nozzle throat the ratio  $w/A$  decreases again and now there may physically exist either value of the  $(p/p_0)$  corresponding to a given  $w/A$  with a continuous variation of pressure through the nozzle being maintained. Thus at section d the pressure may be that corresponding to d or d'. The final pressure distributions that may exist for reversible and maximum mass rate of flow are shown in the sketch below as solid lines.



Suppose now the nozzle to be operating with  $w$  less than maximum. Proceeding in the manner described above the pressure distribution indicated by the dashed line on the sketch is obtained. Since

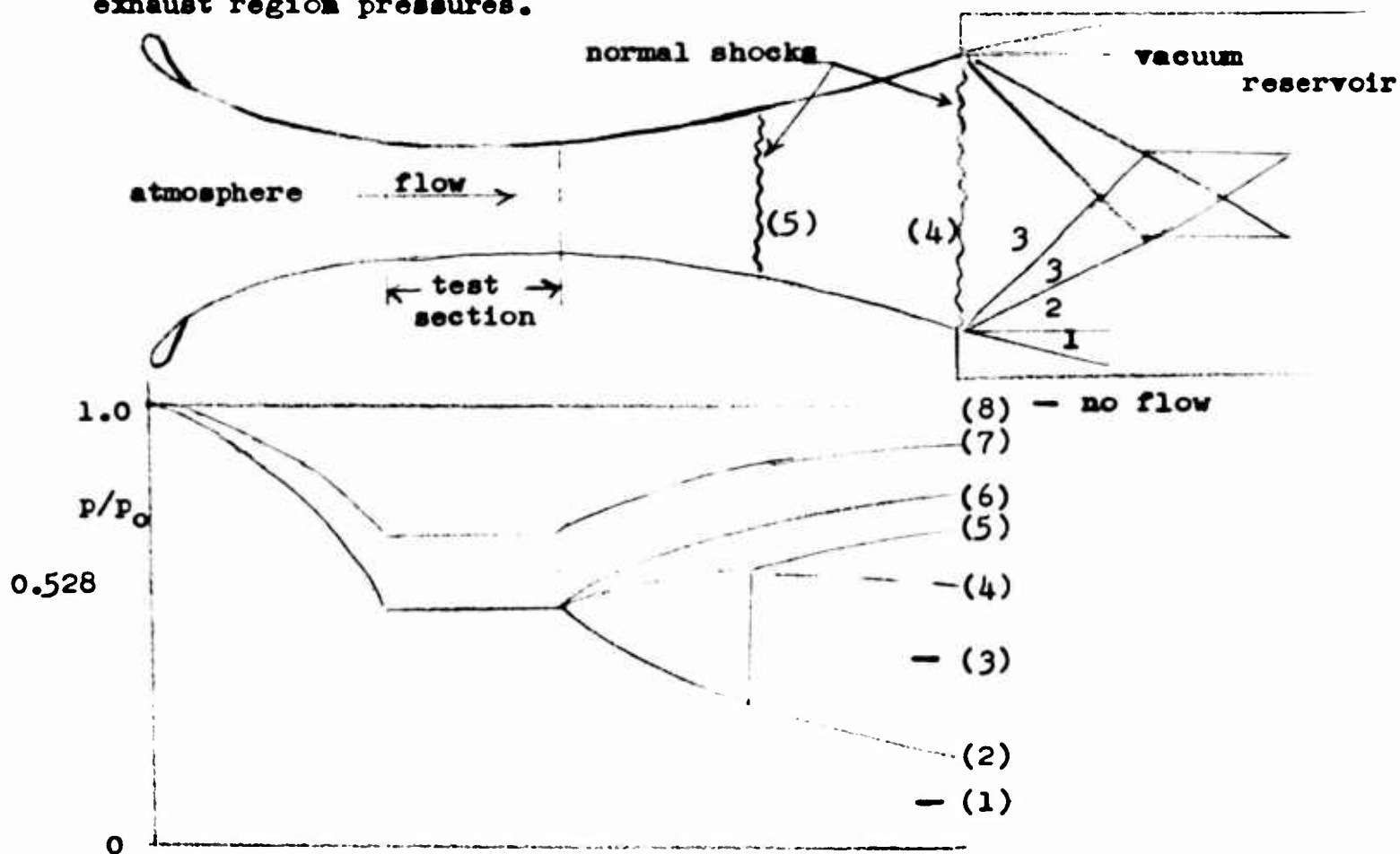
$$M = f\left(\frac{p}{p_0}\right)$$

a Mach number scale may be placed along the vertical ordinate of the graph. This scale indicated in what Mach number range the nozzle is operating.

### Nozzle Flow With Shock in Diffuser

The above example illustrates that for an exhaust region pressure between  $p_f$  and  $p_r$  there is no solution of the relation  $(w/A) = f(p/p_o)$  hence it is impossible to have reversible flow through the nozzle in this range of exhaust region pressures. Physically it is possible to have a discharge region pressure in this range. What happens when such an exit region pressure does exist? To answer this question let us discuss the operating characteristics of a nozzle used as a high speed wind tunnel.

The figure below shows a wind tunnel which operates intermittently by means of an evacuated reservoir. The atmosphere acts as the supply region from which air is drawn through the convergent section, test section, and diffuses into the evacuated reservoir. Below the sketch of the tunnel there is indicated the pressure distributions through the tunnel for seven different exhaust region pressures.



During the operation of the tunnel seven distinct conditions present themselves.\*

\*Paraphrase of pp. 3-5, Part I, High Speed Aerodynamic Lecture Series by Dr. B. H. Goethert.

1. For condition one, wherein the pressure in the reservoir is less than the pressure in the end of the diffuser, the tunnel is operating at maximum rate of flow with subsonic, sonic, and supersonic flow in the convergent, straight, and divergent sections of the nozzle respectively. The transition of the diffuser pressure to the lower reservoir pressure is achieved through a system of expansion waves.

2. For condition two, wherein the pressure in the reservoir has been increased to the diffuser outlet pressure by the inflowing air, no pressure disturbance occurs at the diffuser end.

3. For condition three, wherein the pressure in the reservoir has become greater than the diffuser outlet pressure, the transition to the greater reservoir pressure is produced by an oblique shock wave with flow upstream of nozzle exit unaffected.

4. For condition four the pressure in the reservoir has increased to a value which produces a normal shock wave at the nozzle exit.

5. For condition five the reservoir pressure has attained a value which produces a normal shock in the diffuser. Flow preceding the shock is unaffected. Downstream of the shock subsonic flow exists.

6. For condition six the reservoir pressure has reached a value which produces reversible flow throughout the tunnel with sonic flow in the throat and subsonic flow elsewhere.

7. For condition seven the reservoir pressure has reached a value producing subsonic flow throughout with a reduced mass rate of flow.

Notice that the flow conditions in the test section remain constant as long as the reservoir pressure is not greater than that corresponding to condition six.

The analysis of nozzle flow that we have attempted so far has been confined to reversible flow considerations only. The flow through a discontinuity such as a shock wave is irreversible and hence we can not predict the nozzle pressure distribution such as that corresponding to condition 5 by the analysis we have made so far. In order to complete our study of the operating characteristics of a nozzle we will need to consider plane shock waves. This awaits our further attention.

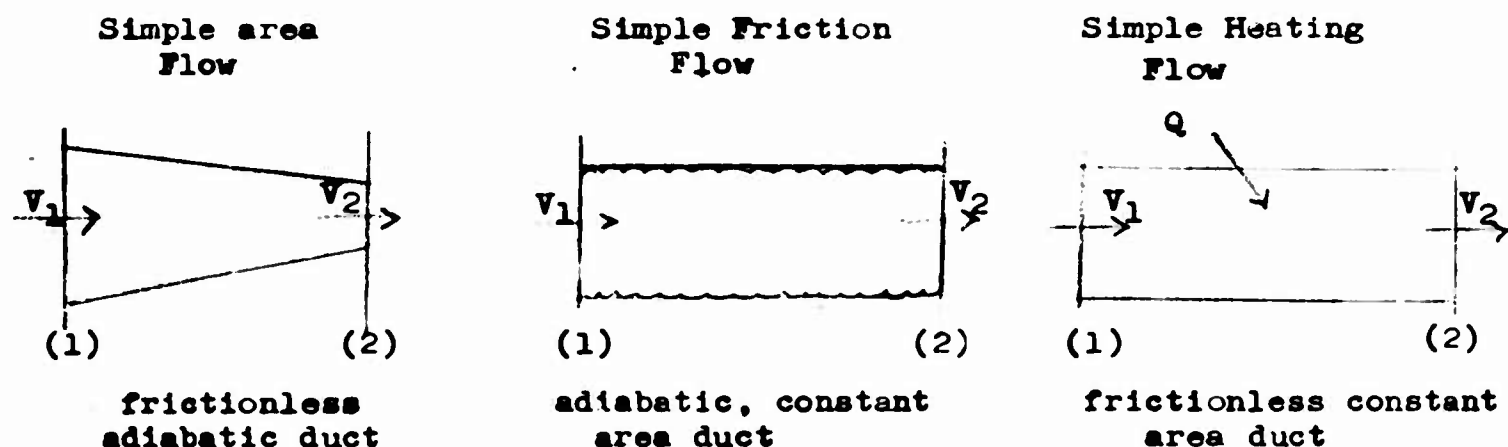
## Lecture Note 10

## SIMPLE AREA FLOW

Consider a gas to be flowing steadily through a duct which satisfies the conditions

- (a) Constant area,
- (b) frictionless, and
- (c) adiabatic.

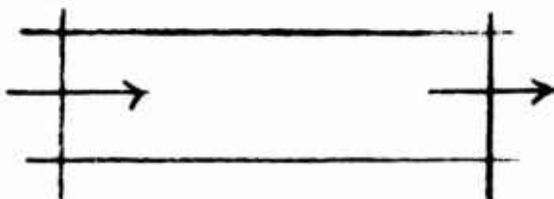
The stream properties of such a flow would in general be constant throughout. If, however, any one of the conditions listed was removed the stream properties would then change with the effect (variable area, friction or heating) present. In the work to follow we will study how the stream properties ( $p$ ,  $p_0$ ,  $T$ ,  $T_0$ ,  $V$ ,  $M$ ,  $F$ ) depend upon each of these effects individually and then collectively. The individual cases to be analyzed are shown in the figure below. Each of the flows indicated in the figure is of practical importance, for example, in the study of flow through a ram-jet engine. The simple area type of flow applies to the inlet diffuser and exhaust



nozzle, the simple heating flow to the combustion chamber and the simple frictional flow to flow between diffuser and flame holders of such an engine.

The purpose of this note is to make a study of the simple area type of flow. Our immediate objective will be to determine how the stream properties ( $p$ ,  $p_0$ ,  $T$ ,  $T_0$ ,  $V$ ,  $M$ ,  $F$ ) of the flow depend upon the independent variable area,  $A$ . A physical interpretation of the problem at hand is, for example, the following. Assume we have a frictionless, adiabatic flow of a perfect gas in a constant area duct. The stream properties of this flow are invariant throughout.

Frictionless, adiabatic



Stream properties constant

Frictionless, Adiabatic



Stream properties may vary with area

Suppose now that we vary at will the area of the duct, that is, the duct area becomes the independent variable of the flow. We want to investigate how the stream properties of the flow change with, or depend upon, the duct area. We may note immediately that the stream properties  $p_0$  and  $T_0$  do not vary with area in the simple area type of flow since it is an isentropic flow. The simple area flow may be used to accelerate a flowing gas in which case the duct used is called a nozzle or conditions may be such that the flow is being decelerated in which instant the duct is called a diffuser.

The following expressions relating the stream properties of the flow under study may be written

$$p = \rho RT \quad (\text{State})$$

$$w = \rho AV \quad \text{constant} \quad (\text{Continuity})$$

$$M^2 = \frac{V^2}{kRT} \quad (\text{Mach No.})$$

$$F = pA(1 + kM^2) \quad (\text{Impulse Function})$$

$$p_0 = p \left(1 + \frac{k-1}{2} M^2\right)^{\frac{k}{k-1}} = \text{constant} \quad (\text{2nd Law})$$

$$T_0 = T + \frac{V^2}{2c_p} = T \left(1 + \frac{k-1}{2} M^2\right) = \text{constant} \quad (\text{1st Law})$$

The variables in this group of six equations are (since  $M$  always appears as a squared quantity, use  $M^2$  as a variable)

$$p, \rho, T, A, V, M^2, F.$$

Selecting one (area) of these seven variables as independent we may determine each of the remaining six dependent variables ( $p, \rho, T, V, M^2, F$ ) in terms of the independent quantity,  $A$ . Thus by assuming  $A$  to be known

we have six equations in six unknowns. To solve the equations as they stand would be difficult if not impossible. It will be found convenient therefore, to first reduce the equations to a set of linear differential equations with variables in logarithmic differential form.

The equation of state becomes

$$\ln p = \ln \rho + \ln T + \ln R$$

differentiating

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

For total pressure we may write

$$\ln p_0 = \ln p + \frac{k}{k-1} \ln \left( 1 + \frac{k-1}{2} M^2 \right)$$

differentiating

$$0 = \frac{dp}{p} + \frac{k}{k-1} \left[ \frac{\frac{k-1}{2} (dM^2)}{1 + \frac{k-1}{2} M^2} \right]$$

$$0 = \frac{dp}{p} + \frac{\frac{kM^2}{2}}{1 + \frac{k-1}{2} M^2} \frac{dM^2}{M^2}$$

Reducing similarly each of the six equations to a differential form and assembling the results there is obtained

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

(State)

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dV}{V} = 0$$

(Continuity)

$$\frac{dM^2}{M^2} = 2 \frac{dV}{V} - \frac{dT}{T}$$

(Mach No.)

$$\frac{dF}{F} = \frac{dp}{p} + \frac{kM^2}{1 + kM^2} \frac{dM^2}{M^2} + \frac{dA}{A}$$

(Impulse Function)



$$\frac{dp}{p} + \frac{\frac{k-1}{2} M^2}{1 + \frac{k-1}{2} M^2} \frac{dM^2}{M^2} = 0 \quad (2nd \text{ Law})$$

$$\frac{dT}{T} + \frac{\frac{k-1}{2} M^2}{1 + \frac{k-1}{2} M^2} \frac{dM^2}{M^2} = 0 \quad (1st \text{ Law})$$

To investigate the variation of the dependent variables with area we must find expressions relating

$$\frac{dM^2}{M^2} \text{ to } \frac{dA}{A}, \frac{dp}{p} \text{ to } \frac{dA}{A}, \text{ etc.}$$

and interpret the results. First obtain  $\frac{dM^2}{M^2}$  in terms of  $\frac{dA}{A}$ . This may be done as follows:

By Mach equation get

$$\frac{dM^2}{M^2} = f \left( \frac{dV}{V}, \frac{dT}{T} \right)$$

using continuity for  $\frac{dV}{V}$  in above find

$$\frac{dM^2}{M^2} = f \left( \frac{d\rho}{\rho}, \frac{dA}{A}, \frac{dT}{T} \right)$$

using equation of state get

$$\frac{dM^2}{M^2} = f \left( \frac{dp}{p}, \frac{dA}{A}, \frac{dT}{T} \right)$$

and with 1st and 2nd Law results obtain

$$\frac{dM^2}{M^2} = f \left( M^2, \frac{dA}{A} \right)$$

or

$$\frac{dM^2}{M^2} = - \left[ \frac{2(1 + \frac{k-1}{2} M^2)}{1 - M^2} \right] \frac{dA}{A}$$

The results of similar simultaneous solutions of the applicable equations are summarized below

	$dA/A$
$\frac{dM^2}{M^2}$	$-\frac{2(1 + \frac{k-1}{2} M^2)}{1 - M^2}$
$\frac{dV}{V}$	$-\frac{1}{1 - M^2}$
$\frac{dp}{p}$	$\frac{kM^2}{1 - M^2}$
$\frac{d\rho}{\rho}$	$\frac{M^2}{1 - M^2}$
$\frac{dT}{T}$	$\frac{(k-1) M^2}{1 - M^2}$
$\frac{dF}{F}$	$\frac{1}{1 + kM^2}$
<p>Table is read as follows</p> $\frac{dV}{V} = \left( -\frac{1}{1 - M^2} \right) \frac{dA}{A} \text{ etc.}$	

General conclusions can be made relative to the variation of the stream properties of the flow with the independent variable area by these relations. For example

$$\frac{dV}{V} = - \frac{1}{1 - M^2} \frac{dA}{A}$$

indicates that in a subsonic flow ( $M < 1$ ) a convergent passage ( $dA < 0$ ) will accelerate the flow ( $dV > 0$ ). Conversely in a supersonic flow a diverging passage is required to accelerate the flow. Similar reasoning may be applied to determine the manner in which the remaining stream properties vary with the duct area in a subsonic or supersonic flow.

### Stream properties as functions of Mach Number

For the isentropic flow under consideration analytical relations may be found between each variable of the flow and the flow Mach Number by rearrangement and integration of the tabulated results above. As an example let us integrate the equation relating Mach Number to area between the point in the flow where  $M^2 = 1$  and  $A = (A)_{M=1}$  and any general point where  $M^2 = M^2$ . Denoting  $(A)_{M=1}$  by  $A^*$  we have

$$\int_{A^*}^A \frac{dA}{A} = - \int_1^{M^2} \frac{1 - M^2}{2(1 + \frac{k-1}{2} M^2)} \frac{dM^2}{M^2}$$

Letting  $M^2 = x$  and  $\frac{k-1}{2} = b$  and multiplying through by -2, there follows

$$\begin{aligned} -2 \ln \frac{A}{A^*} &= \int_1^x \frac{1 - x}{1 + bx} \frac{dx}{x} \\ &= - \int_1^x \frac{dx}{1 + bx} + \int_1^x \frac{dx}{(1 + bx)x} \end{aligned}$$

which expands, by partial sums, into

$$\begin{aligned}
 -2 \ln \frac{A}{A^*} &= - \int_1^x \frac{dx}{1+bx} + \int_1^x \frac{dx}{x} - b \int_1^x \frac{dx}{1+bx} \\
 &= -(b+1) \int_1^x \frac{dx}{1+bx} + \int_1^x \frac{dx}{x} \\
 &= -\frac{b+1}{b} \ln \frac{1+bx}{1+b} + \ln x
 \end{aligned}$$

$$-2 \ln \frac{A}{A^*} = - \ln \left[ \frac{1}{x} \left( \frac{1+bx}{1+b} \right)^{\frac{b+1}{b}} \right]$$

Whence

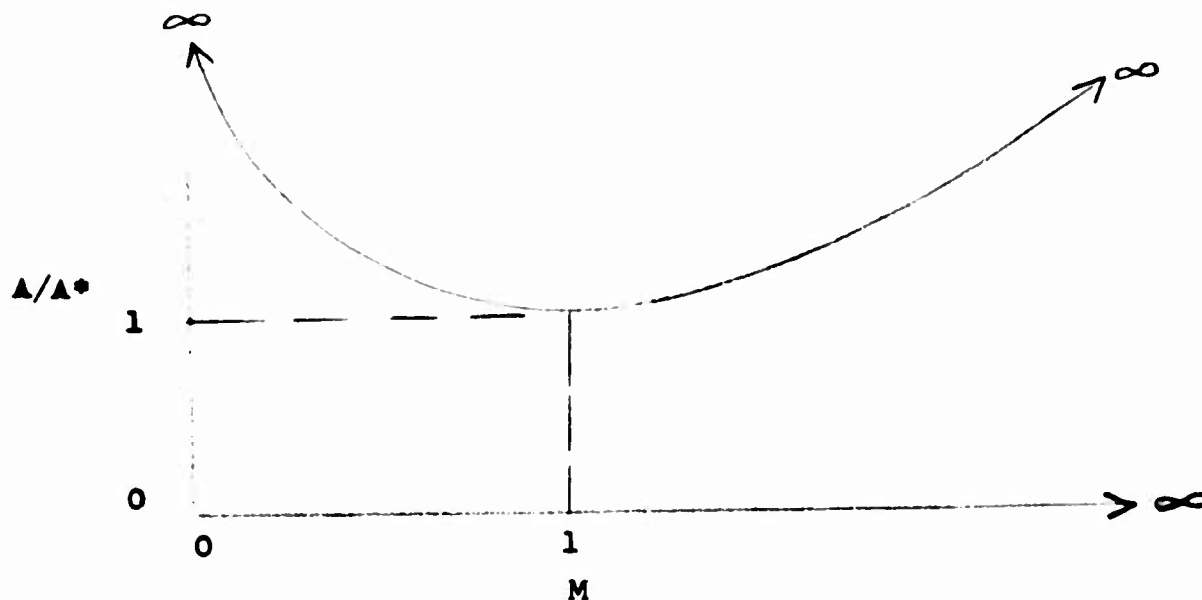
$$\left( \frac{A}{A^*} \right)^2 = \frac{1}{M^2} \left[ \frac{1 + \frac{k-1}{2} M^2}{\frac{k+1}{2}} \right]^{\frac{k+1}{k-1}}$$

or, finally

$$\frac{A}{A^*} = \frac{1}{M} \sqrt{\left[ \frac{2(1 + \frac{k-1}{2} M^2)}{k+1} \right]^{\frac{k+1}{k-1}}}$$

which gives the ratio of the local flow area to flow area for  $M=1$  as a function of the local Mach Number for an isentropic flow.

This relation is given below as a curve of  $A/A^*$  versus  $M$  for a given value of  $k$ . The curve may be interpreted for an assumed  $A^*$  as the flow area required



for a given Mach Number. It serves to illustrate further the area variation required to increase the Mach Number of a flow from sub to supersonic values. This curve or the tabulated values of  $A/A^*$  vs.  $M$  (Tables 30 through 35 Keenan and Kaye) may be used to find the area change required for any reversible Mach Number change. For example to diffuse air from  $M_1 = 0.8$  to  $M_2 = 0.25$  would require a diffuser with an area ratio of

$$\frac{A_2}{A_1} = \frac{(A/A^*)_2}{(A/A^*)_1} = \frac{2.40}{1.038} = 2.31$$

Or to expand from  $M_1 = 0.25$  to  $M_2 = 2.4$  with an inlet area of  $A_1 = 3 \text{ in}^2$ . would require a throat area of

$$A_{th} = A^* = \frac{1}{(A/A^*)_1} A_1 = \frac{1}{2.40} 3 = 1.25 \text{ in}^2.$$

and an exit area of

$$A_2 = \frac{(A/A^*)_2}{(A/A^*)_1} A_1 = \frac{2.40}{2.40} 3 = 3 \text{ in}^2$$

To continue finding the stream properties as functions of the flow Mach Number consider obtaining the relationship between  $V/V^*$  and  $M$ . By previous results

$$\frac{dV}{V} = - \frac{1}{1 - M^2} \frac{dA}{A}$$

and using

$$\frac{dA}{A} = - \frac{1 - M^2}{2(1 + \frac{k-1}{2} M^2)} \frac{dM^2}{M^2}$$

we get

$$\int_{V^*}^V \frac{dV}{V} = \int_1^{M^2} \frac{1}{2(1 + \frac{k-1}{2} M^2)} \frac{dM^2}{M^2}$$

which becomes

$$M^* = \frac{V}{V^*} = M \sqrt{\frac{k}{2(1 + \frac{k-1}{2} M^2)}}$$

Proceeding along the same lines the following relation is obtained for isentropic flow with variable area

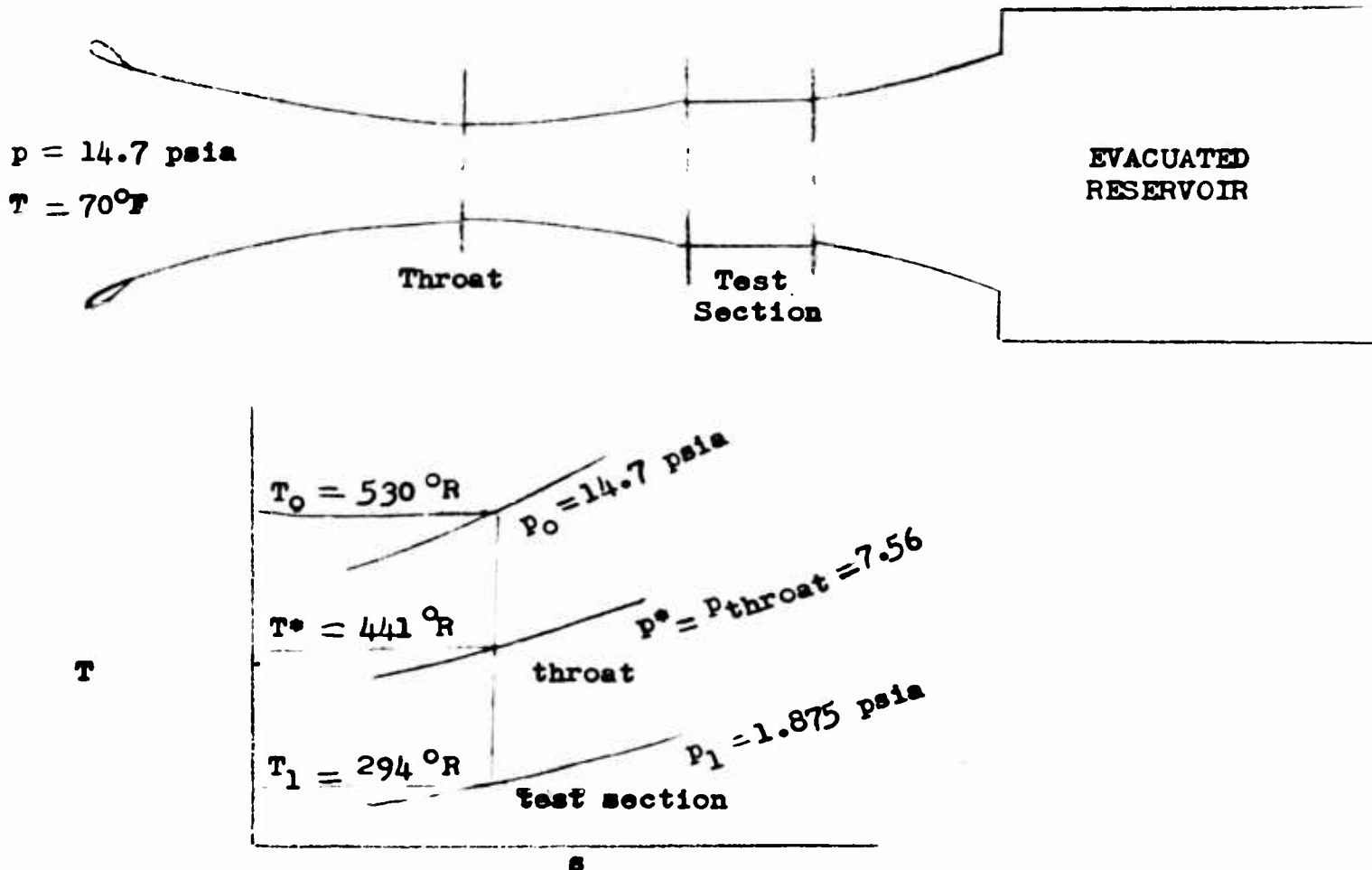
$$\frac{F}{F^*} = \frac{1}{M} \sqrt{\frac{k M^2}{2(k+1)(1 + \frac{k-1}{2} M^2)}}$$

Similarly  $p/p^*$ ,  $T/T^*$ ,  $\rho/\rho^*$  may be found in terms of Mach Number. However, these ratios do not prove as useful† in applications as the ratios  $p/p_0$ ,  $T/T_0$ ,  $\rho/\rho_0$  which have been given as functions of Mach Number in lecture note 2

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† They are not therefore derived here nor tabulated in the gas tables.

As an example of the use of Table 30 of Gas Tables by Keenan and Kaye suppose an intermittent supersonic wind tunnel exhausting to a vacuum reservoir is to be designed for a Mach No. of 2 in a  $1.2 \text{ ft}^2$  test section. If the tunnel receives atmospheric air at  $p = 14.7 \text{ psia}$  and  $T = 70^\circ\text{F}$ , what are the required tunnel throat area and the test section stream properties? Assume isentropic flow with  $k = 1.4$ .



A schematic diagram of the tunnel and the flow process on a  $T-s$  graph are given in the figures above. With  $M = 2$  in the test section find

$$(M^*)_1 = \left(\frac{V}{V^*}\right)_1 = 1.633 \quad \text{where subscript 1 stands for test section}$$

$$\left(\frac{A}{A^*}\right)_1 = 1.6875; \left(\frac{T}{T_0}\right)_1 = 0.555$$

$$\left(\frac{\rho}{\rho_0}\right)_1 = 0.23; \left(\frac{p}{p_0}\right)_1 = 0.1278$$

Therefore

$$A_{th} = A^* = \frac{1}{(A/A^*)_1} A_1 = \frac{1.2}{1.6875} = 0.712 \text{ ft}^2$$

$$T_1 = \left( \frac{T}{T_0} \right)_1 T_0 = 0.555(530) = 294^\circ\text{R} (-166^\circ\text{F})$$

$$p_1 = \left( \frac{p}{p_0} \right)_1 p_0 = 0.1278(14.7) = 1.875 \text{ psia}$$

$$\rho_1 = \left( \frac{\rho}{\rho_0} \right)_1 \rho_0 = 0.23 \frac{14.7 \times 144}{1715 \times 530} = 0.000535 \frac{\text{slugs}}{\text{ft}^3}$$

$$V_1 = M_1 \sqrt{\gamma R T_1} = 2 \sqrt{1.4(1715)(294)} = 2 \times 840 = 1680 \text{ ft/sec.}$$

$$w = \rho_1 A_1 V_1 = (0.000535)(1.2)(1680) = 1.08 \frac{\text{slugs}}{\text{sec.}}$$



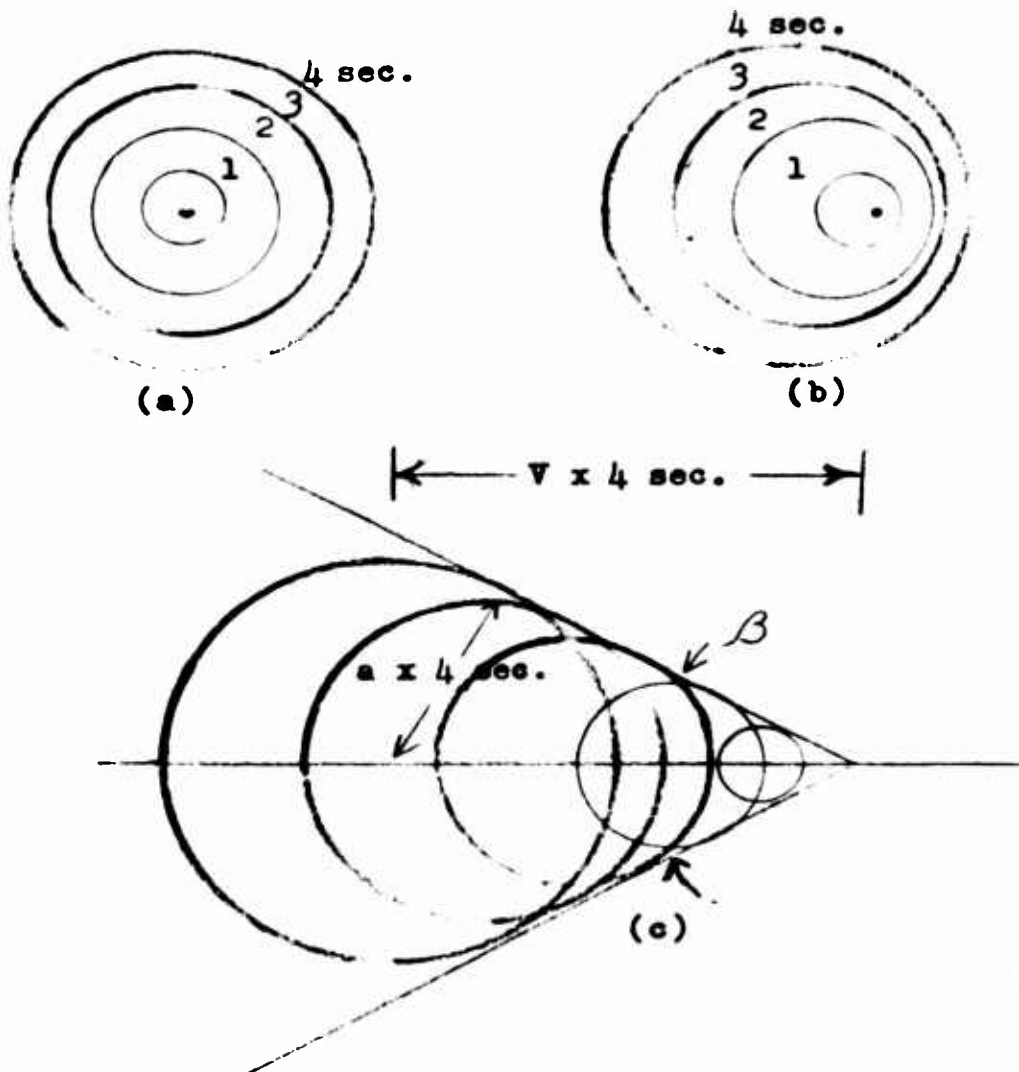
## Lecture Note 11

## COMPRESSIBILITY PHENOMENA

The shock waves occurring in a nozzle as discussed previously are related to the flow discontinuities existing about bodies in supersonic flow fields. In order to establish relationships for the shock phenomena occurring in a supersonic nozzle and also to analyze in general the discontinuities occurring in supersonic flow we take up next the study of wave propagation and compressibility phenomena in a compressible fluid.

Consider an infinitely small point source of disturbance which may produce periodic disturbances that propagate with the speed of sound through the surrounding medium. Let this point source exist in a stationary fluid field under the following conditions

- (a) point source stationary
- (b) point source moving at subsonic velocity
- (c) point source moving at supersonic velocity



The waves emanating periodically from the source will propagate spherically outward forming the wave patterns indicated in the sketches herewith after any given time interval. When the fluid and point source are at rest concentric circles are formed by the wave pattern. When the source moves at a velocity less than the wave propagation velocity, the waves form circles about their point of origin and consequently are no longer concentric since the source emanates each wave from a different position in the fluid. In case (c) the spherical wave fronts are formed in such a manner that all wave front circles are tangent to a line making an angle  $\beta$  with the direction of the source velocity such that

$$\sin \beta = \frac{1}{M} .$$

The tangent line is called a Mach line (each point on the Mach line is traveling normal to the line at the velocity of sound - hence Mach line is a sound wave and not to be confused with a shock wave which has propagation velocity greater than speed of sound).

In case (b) it is observed that in the absence of fluid viscosity effects the disturbance waves will not die out and will influence the fluid field at an infinite distance about the source as time progresses. In case (c) however it should be understood that the fluid field is completely undisturbed forward of the Mach cone and only within the cone are disturbances experienced by the fluid.

#### Subsonic Motion of a Wing

The above considerations can be applied to the steady motion of a rigid body of finite size through a fluid by imagining the steady state motion of the object to be acquired through a series of small separate impulses. Each impulse giving rise to an increase in velocity (an acceleration) and causing a pressure disturbance to emanate from the body. Each of these disturbances spread out from the object with the speed of sound and in subsonic motion would produce a flow pattern extending, ideally, an infinite distance from the body. Actually, of course, the disturbance fields about the object would die out at some distance from the object due to fluid viscosity effects.

Let us apply these ideas to the subsonic motion of an airfoil in a fluid at rest (the atmosphere). With the wing moving at a steady subsonic velocity, there exists about it a region of disturbance which is characterized by values of pressure, density, and velocity different from the free stream values of these properties. This disturbance field may be imagined to have been produced by emanation of waves from the airfoil during its acceleration up to the final steady state velocity. After the disturbance field (stream conditions about this airfoil) has been established in this manner it will persist until the wing is again accelerated and new waves sent out. As long as the stream velocity relative to the wing is everywhere subsonic, these waves will radiate in all directions. Thus the pressure and velocity distributions about a wing in steady subsonic motion are continuous, i. e., no sudden changes in pressure or velocity exist as through a discontinuity such as a shock wave.

Supersonic Motion of an Airfoil

Suppose now an airfoil to be accelerated to a low supersonic velocity by a series of separate impulses. When the wing attains a supersonic velocity the disturbance field about it cannot extend to great distances since the wing tends to overtake the disturbances it propagates. However, immediately in front of the wing there must be some disturbance characterized by the stream lines spreading out so as to enclose the body. Thus we are led to conclude that a disturbance due to the wing extends some finite distance ahead of it. Since this distance is finite there must be a sudden change in stream properties at the boundary between this disturbance field and the free stream fluid. These circumstances give rise, therefore, to a discontinuity and the existence of a discontinuous pressure and velocity distribution about the wing. This discontinuity is known as a shock wave and through the shock wave there are sudden changes in the stream properties. In steady supersonic motion of a wing this discontinuity remains at a fixed distance from the wing and propagates, therefore, into the free stream fluid at the speed of the wing.

Some insight into the origin and nature of the wave discontinuity present in the supersonic motion of objects can be obtained by considering the following facts which will not be validated here.

- (a) The velocity of wave propagation in a fluid is a function of the pressure rise across the wave and is given by

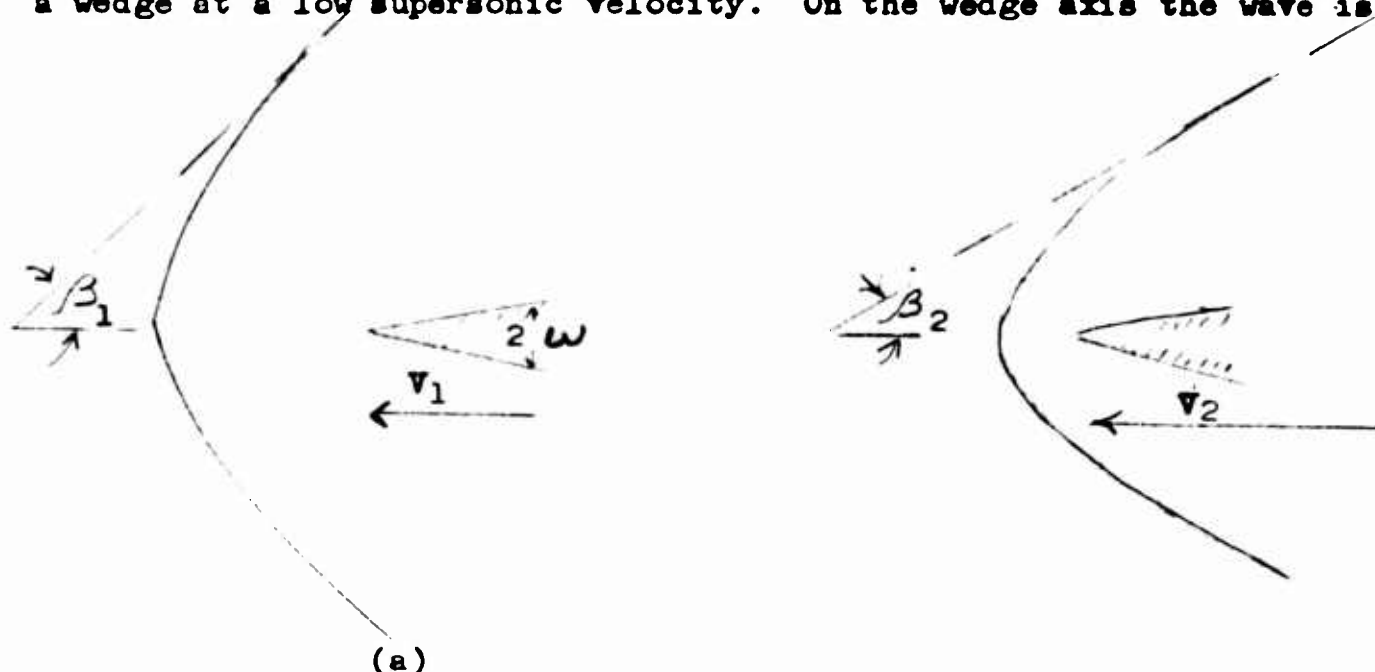
$$V_{\text{propag.}} = \sqrt{kRT_1 \left[ \frac{k-1}{2k} \left( \frac{p_2}{p_1} \right) + \frac{k-1}{2k} \right]}$$

where subscripts 1 and 2 refer to conditions upstream and downstream of the wave respectively. (For  $p_2/p_1 = 1$  observe that  $V = \sqrt{kRT}$  and hence the pressure increment across a sound wave must be infinitesimal).

- (b) A pressure pulse moves at sonic speed with respect to the fluid immediately in front of it.
- (c) The fluid in the wave of a positive pressure pulse is left with a disturbance velocity in the same direction as the pulse movement.

As a consequence of these facts it follows that in a series of positive pressure pulses each pulse overtakes ones in front resulting in a coalescence of the waves into a strong wave with a finite pressure rise. The resultant strong wave propagates at a supersonic velocity. Now as a wing is accelerated to a supersonic velocity the pulses sent out by the wing during its acceleration coalesce to form a strong wave in front of the wing. During the formation of this wave its velocity increases until finally its propagation velocity becomes equal to the wing velocity after which time it remains at a fixed and finite distance in front of the wing.

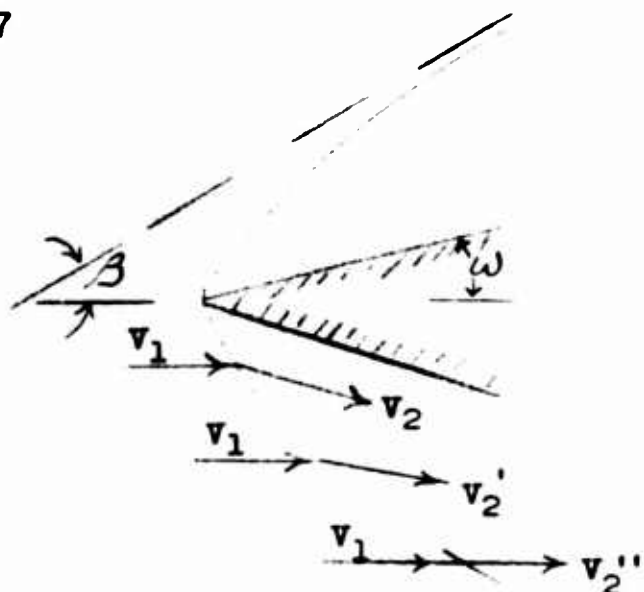
In the figure there is indicated the form of such a wave that would accompany a wedge at a low supersonic velocity. On the wedge axis the wave is normal to



the relative velocity of the stream. As we go outward from the axis the wave becomes weaker, these portions of the wave being further from the source disturbance. In accordance with (a) above these parts of the wave propagate at a lower velocity. Consequently as we go away from the axis the wave bends backward approaching asymptotically in a straight line making the Mach angle  $\beta$ , with the axis. At higher Mach numbers the wave is closer to the wedge as indicated in the sketch where  $M_2 > M_1$ .

Consider the transition of conditions about the wedge from the steady state condition (a) to the steady state condition (b) figure above. The wedge may be imagined to be accelerated from  $M_1$  to  $M_2$  by a series of impulses. As a result of any given impulse the velocity of the wedge is increased and a pulse is sent out. During the time interval required for the pulse to travel from the wedge to the wave the velocity of the wedge is greater than that of the wave and the wedge moves closer to the wave. After the pulse reaches the wave it causes an increase in the pressure rise across the wave and in the wave velocity. Near the axis of the wedge the wave travels with the new velocity of the wedge and away from the axis it gradually weakens and curves backward making the Mach angle  $\beta_2$  with the axis. Notice that  $\beta_2 < \beta_1$  in accordance with  $\sin \beta = \frac{1}{M}$ .

If the Mach number of the wedge is sufficiently great and the half wedge angle less than  $45^\circ$  the wave will attach itself to the wedge as shown below.



The relative velocities of the stream through the wave at various points on the wave are shown in the figure. Close in to the wedge where the wave is strongest the stream velocity is deflected an angle  $\omega$ . At points farther from the wedge the velocity is affected less and less as the wave becomes weaker and weaker.

Shock waves not normal to the free stream velocity and through which the stream velocity is deflected are called oblique shock waves. The Mach number of the stream entering an oblique wave is supersonic while the leaving stream Mach number may be supersonic or subsonic depending upon the angle  $\omega$  and the inlet Mach Number. Normal shock waves are normal to the free stream and leaving stream velocities. The Mach number of the fluid passing through a normal shock always changes from supersonic to subsonic.

A description of the flow field about objects in subsonic and supersonic flight and the manner in which this field is built up has been attempted in this note. Two general features in particular that have been discussed and that should be emphasized are restated here.

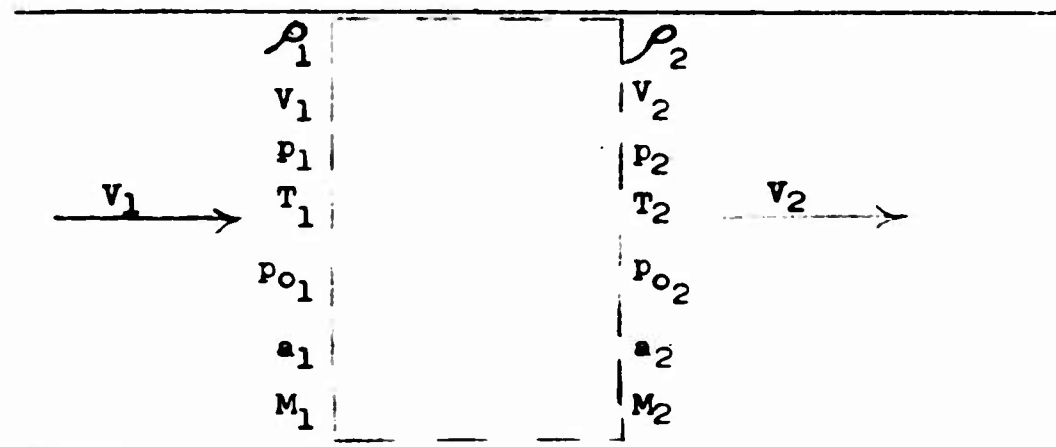
- (a) The pressure and velocity distributions about a body in subsonic motion are continuous and extend to great distances from the body.
- (b) The flow field about an object in supersonic motion extends to finite distances in some directions - the boundary of the field in these directions being marked by a discontinuity (a shock wave) in the field stream properties.

Finally, it is pointed out again that as a shock wave becomes weaker it becomes in the limit a sound wave.

## Lecture Note 12

## NORMAL SHOCK WAVE

Imagine a fluid to be flowing through an adiabatic, frictionless, constant area duct as shown below. In this steady flow field consider the region bounded by the dashed line. We refer to this region as the control volume and its boundary as the control surface. Subscripts 1 and 2 designate flow conditions at the control volume inlet and exit respectively. Let us investigate



this problem: For given inlet conditions to the control volume what are the possible exit conditions? One obvious and yet not trivial situation is that in which the inlet and exit conditions are identical. There is possible, however, a not so obvious situation in which the exit conditions may differ from those at the inlet if the inlet flow is supersonic. When this situation exists and the control region's thickness is very small (of order  $10^{-8}$  inches for air) the region represents what is called a normal shock wave. To determine, for given inlet conditions to the shock, the seven variables listed in the figure at the exit of the control volume requires the simultaneous solution of seven equations. The equations are obtained from application of the following definitions and physical laws.

- (1) Conservation of Mass (continuity equation)
- (2) Newton's 2nd Law of Motion (momentum equation)
- (3) 1st Law of Thermo (energy equation)
- (4) Equation of state
- (5) 2nd Law of Thermodynamics
- (6) Velocity of sound
- (7) Mach Number definition

The equations to be solved are

$$\begin{aligned}
 (1) \quad \rho_1 v_1 &= \rho_2 v_2 & (dA &= 0) \\
 (2) \quad p_1 + \rho_1 v_1^2 &= p_2 + \rho_2 v_2^2 & (dA = \mathcal{F} = dF = 0)
 \end{aligned}$$

$$(3) \quad c_p T_1 + \frac{v_1^2}{2} = c_p T_2 + \frac{v_2^2}{2} \quad (dT_\theta = 0)$$

$$(4) \quad \frac{p_1}{\rho_1 T_1} = \frac{p_2}{\rho_2 T_2} \quad (p = \rho RT)$$

$$(5) \quad p_{o1} \left( \frac{p_1}{\rho_1} \right)^{\frac{1}{k-1}} = p_{o2} \left( \frac{p_2}{\rho_2} \right)^{\frac{1}{k-1}} \quad (ds = \frac{dq}{T})$$

$$(6) \quad \frac{a_1^2}{T_1} = \frac{a_2^2}{T_2} \quad (a^2 = kRT)$$

$$(7) \quad \frac{M_1^2 T_1}{v_1^2} = \frac{M_2^2 T_2}{v_2^2} \quad (M^2 = \frac{v^2}{kRT})$$

Equations (1) through (4) represent 4 equations in 4 unknowns and were first solved for the non-obvious condition of  $\rho_1 \neq \rho_2$ , etc. by Rankine and Hugoniot to obtain what are now called the Rankine-Hugoniot relations. They will not be derived here, however. With the following goals in mind it is necessary to effect a solution of equations (1) through (7). Our goals are the following relations\*\*

$$\frac{p_2}{p_1} = f_1(M_1)$$

$$\frac{\rho_2}{\rho_1} = f_2(M_1)$$

$$\frac{T_2}{T_1} = f_3(M_1)$$

$$\frac{p_{o2}}{p_{o1}} = f_4(M_1)$$

$$M_2^2 = f_5(M_1)$$

$$\frac{p_{o2}}{p_1} = f_6(M_1).$$

Equation (2) divided by (1) and rearranged gives

$$\frac{p_2}{\rho_2 v_2} - \frac{p_1}{\rho_1 v_1} = v_1 - v_2$$

\*\* Tabulated in "Gas Tables" Tables 48-52

\* See pp. 38-39, Liepmann and Puckett, Aerodynamics of Compressible Fluid, Wiley, 1947.

Using (6) in the form  $kRT = k \frac{p}{\rho} = a^2$  we substitute  $\frac{a^2}{k}$  for  $\frac{p}{\rho}$  and get

$$\frac{a_2^2}{V_2} - \frac{a_1^2}{V_1} = k(V_1 - V_2). \quad (1)$$

Equations (3) and (6) along with the definition of starred quantities give the following relation

$$\frac{a^2}{k-1} + \frac{V^2}{2} = a^{*2} \frac{k+1}{2(k-1)}$$

or

$$a^2 = \frac{1}{2} \left[ (k+1)a^{*2} - (k-1)V^2 \right]. \quad (11)$$

Substituting in (1) the value of  $a_2^2$  and  $a_1^2$  as found from (11) we get

$$\frac{\frac{1}{2} \left[ (k+1)a^{*2} - (k-1)V_2^2 \right]}{V_2} - \frac{\frac{1}{2} \left[ (k+1)a^{*2} - (k-1)V_1^2 \right]}{V_1} = k(V_1 - V_2).$$

this simplifies to

$$\frac{1}{2} (k+1)a^{*2} \frac{(V_1 - V_2)}{V_1 V_2} = \frac{k+1}{2} (V_1 - V_2)$$

whence

$$V_1 V_2 = a^{*2} \quad (111)$$

#### Pressure ratio across a normal shock

Using (1) equation 2 can be written

$$p_2 = p_1 + \rho_1 V_1 (V_1 - V_2)$$

or

$$\frac{p_2}{p_1} = 1 + \frac{\rho_1}{p_1} (V_1^2 - V_1 V_2).$$



From (iii), (ii), and (7)

$$v_1 v_2 = a^2 = a_1^2 \left[ \frac{k-1}{k+1} M_1^2 + \frac{2}{k+1} \right] \quad (iv)$$

If this relation along with  $\frac{p_1}{\rho_1} = \frac{a_1^2}{k}$  is used in the pressure ratio relation there results

$$\frac{p_2}{p_1} = 1 + k \left( M_1^2 - \frac{k-1}{k+1} M_1^2 - \frac{2}{k+1} \right)$$

or

$$\boxed{\frac{p_2}{p_1} = \frac{2k}{k+1} M_1^2 - \frac{k-1}{k+1} = f_1(M_1)}$$

#### Density ratio across a normal shock

From equation (1)

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{v_1^2}{v_1 v_2}$$

Using equation (iv) for  $v_1 v_2$  we have

$$\frac{\rho_2}{\rho_1} = \frac{M_1^2}{\frac{k-1}{k+1} M_1^2 + \frac{2}{k+1}}$$

or

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{1}{\frac{k-1}{k+1} + \frac{2}{(k+1) M_1^2}} = f_2(M_1)}$$

#### Temperature ratio across a normal shock.

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \frac{\rho_1}{\rho_2} = \frac{f_1(M_1)}{f_2(M_1)} = f_3(M_1)$$

Which, after a few tries, becomes

$$\frac{T_2}{T_1} = \frac{\left(\frac{2k}{k-1} M_1^2 - 1\right) \left(1 + \frac{k-1}{2} M_1^2\right)}{\frac{(k+1)^2}{2(k-1)} M_1^2} = f_3(M_1)$$

Total pressure ratio across a normal shock

We have

$$\frac{ds}{c_p} = \frac{dT}{T} - \frac{k-1}{k} \frac{dp}{p} = \frac{dT_0}{T_0} - \frac{k-1}{k} \frac{dp_0}{p_0}.$$

Whence, across a normal shock wave with  $dT_0 = 0$ ,

$$\frac{dp_0}{p_0} = \frac{dp}{p} - \frac{k}{k-1} \frac{dT}{T}.$$

This may be written in terms of  $p$  and  $\rho$  by replacing  $\frac{dT}{T}$  with

$$\frac{dT}{T} = \frac{dp}{p} - \frac{d\rho}{\rho},$$

which follows from the equation of state. Thus

$$\frac{dp_0}{p_0} = \frac{k}{k-1} \frac{d\rho}{\rho} - \frac{1}{k-1} \frac{dp}{p}.$$

Integrating from 1 to 2 we have

$$\ln \frac{p_{02}}{p_{01}} = \ln \left( \frac{\rho_2}{\rho_1} \right)^{\frac{k}{k-1}} - \ln \left( \frac{p_2}{p_1} \right)^{\frac{1}{k-1}}$$

or

$$\frac{p_{02}}{p_{01}} = \left[ \frac{\left( \frac{\rho_2}{\rho_1} \right)^k}{\left( \frac{p_2}{p_1} \right)} \right]^{\frac{1}{k-1}}$$

for

$$\frac{p_{o2}}{p_{o1}} = 1$$

this relation gives

$$\frac{\rho_2}{\rho_1} = \left( \frac{p_2}{p_1} \right)^{1/k}$$

showing that decrease of  $p$  is measure of degree of irreversibility associated with the shock or is a measure of the departure of the process through the shock from a reversible adiabatic process. The total pressure relation, after using

$$\left( \frac{\rho_2}{\rho_1} \right) = f_2(M_1) \quad \text{and} \quad \frac{p_2}{p_1} = f_1(M_1),$$

takes the form

$$\frac{p_{o2}}{p_{o1}} = \left[ \frac{1}{\left( \frac{k-1}{k+1} + \frac{2}{(k+1) M_1^2} \right)^k \left( \frac{2k}{k+1} M_1^2 - \frac{k-1}{k+1} \right)} \right]^{\frac{1}{k-1}}$$

or

$$\frac{p_{o2}}{p_{o1}} = \left[ \frac{\frac{k+1}{2} M_1^2}{1 + \frac{k-1}{2} M_1^2} \right]^{\frac{k}{k-1}} \left[ \frac{2k}{k+1} M_1^2 - \frac{k-1}{k+1} \right]^{\frac{1}{1-k}} = f_4(M_1)$$

#### Mach Number before and after a normal shock

In the derivation of

$$\frac{p_2}{p_1} = f_1(M_1)$$

we could have, by interchanging indices, obtained

$$\frac{p_1}{p_2} = f_1(M_2).$$

Thus with the relation

$$\frac{p_1}{p_2} = \frac{1}{p_2/p_1}$$

we may determine

$$M_2 = f_5(M_1)$$

from

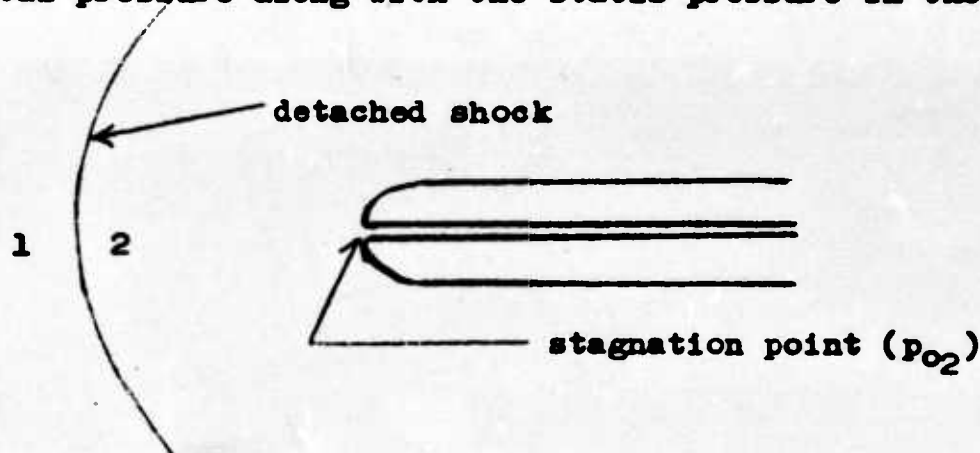
$$\frac{2k}{k+1} M_2^2 - \frac{k-1}{k+1} = \frac{1}{\frac{2k}{k-1} M_1^2 - \frac{k-1}{k+1}}$$

this reduces to the form

$$M_2^2 = \frac{M_1^2 + \frac{2}{k-1}}{\frac{2k}{k-1} M_1^2 - 1} = f_5(M_1)$$

### Rayleigh's Pitot Equation

If a pitot tube is placed in a supersonic flow it will produce a detached shock wave and will measure the total pressure behind a normal shock. This value of total pressure along with the static pressure in the supersonic flow provide



sufficient data to determine the Mach Number of the supersonic flow. The pressure rise, as the flow is brought to rest, is divided into two parts, one  $p_2/p_1$  due to the shock and the other  $p_{02}/p_2$  due to isentropic compression between the shock wave and the pitot tube head. We may write

$$\frac{p_{02}}{p_1} = \frac{p_{01}}{p_1} \cdot \frac{p_{02}}{p_{01}} = \left(1 + \frac{k-1}{2} M_1^2\right)^{\frac{k}{k-1}} \cdot f_4(M_1)$$

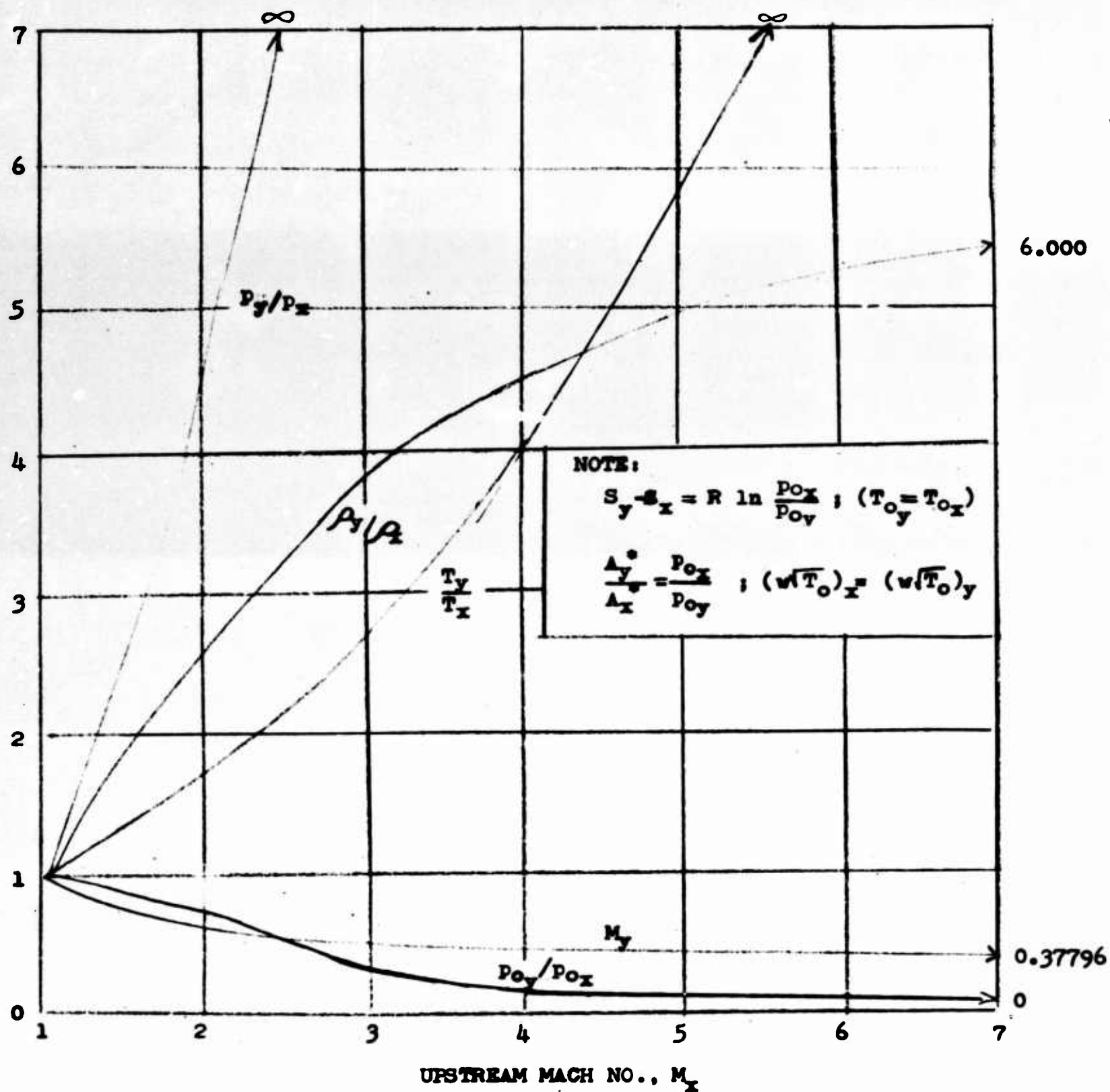
or

$$\frac{p_{02}}{p_1} = \left[\frac{k+1}{2} M_1^2\right]^{\frac{k}{k-1}} \left[\frac{2k}{k+1} M_1^2 - \frac{k-1}{k+1}\right]^{\frac{1}{1-k}} = f_6(M_1)$$

This equation which relates the observed total pressure and the free stream static pressure is known as Rayleigh's pitot equation.

TO ACCOMPANY NOTE 12, NORMAL SHOCK WAVE

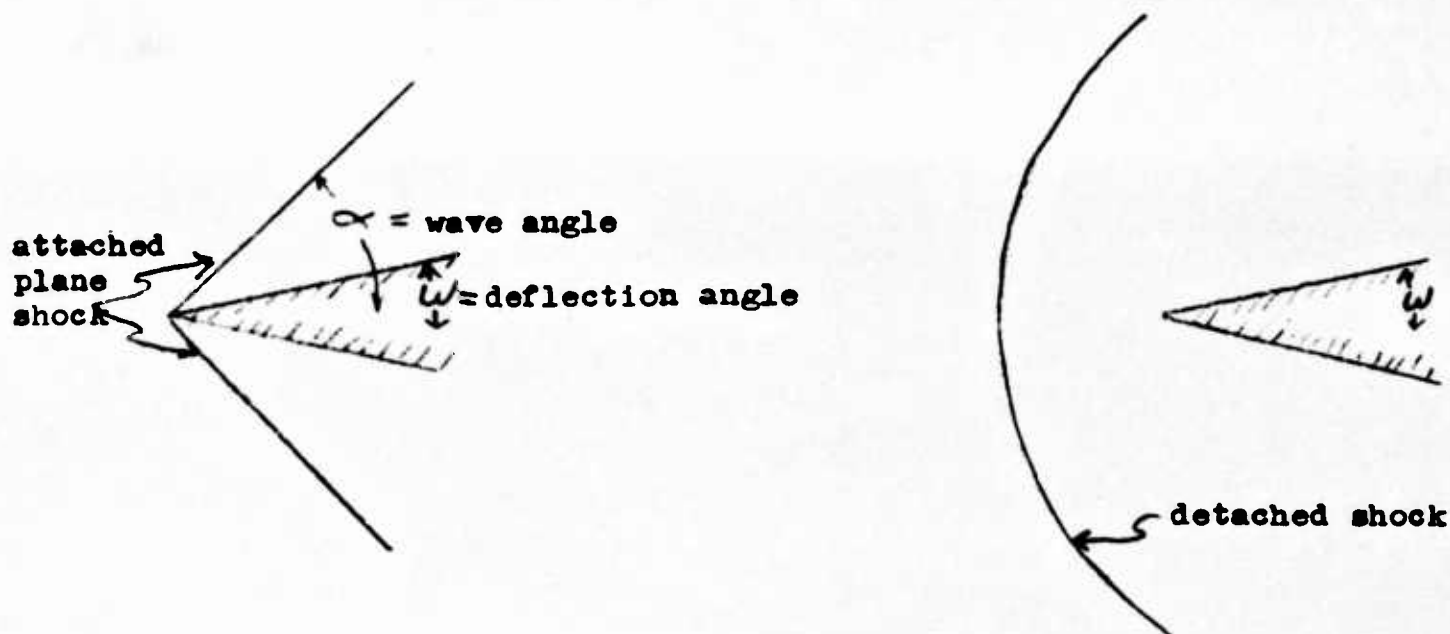
NORMAL SHOCK FUNCTIONS FOR AIR  
(DATA FROM KEENAN & KAYE "GAS TABLES," TABLE 48)



## Lecture Note 13

OBLIQUE SHOCK WAVE

Experiments show that when a wedge shaped object is placed in a supersonic flow there may result a plane shock wave emanating from the nose of the body or there may arise a detached shock wave which is curved and passes in front of the object. It is found that the flow Mach Number and the wedge angle  $\omega$  together determine which of these two types of shocks will occur. Consider the analysis



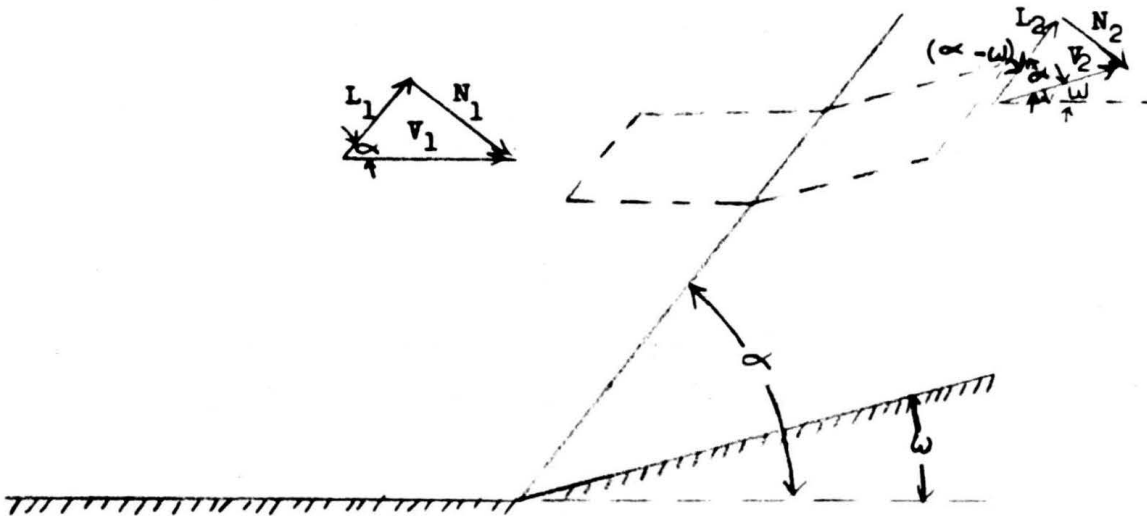
of oblique shock waves with the following purposes in mind:

- (1) To determine the exit conditions from an attached oblique shock wave given the inlet conditions and either  $\omega$  or the wave angle  $\alpha$ .
- (2) To determine the limitations on  $M$  and  $\omega$  for an attached shock to occur.
- (3) To show that the normal shock wave is a special case of the oblique shock with  $\alpha = 90^\circ$  and  $\omega = 0^\circ$ .

In the figure below a flow is deflected through an angle  $\omega$  as it passes through a shock wave which makes an angle  $\alpha$  with the upstream flow velocity.

$N$  = Vel. component normal to wave front

$L$  = Vel. component parallel to wave front



The control volume indicated by the dashed lines is selected such that its upper and lower sides are coincident with the flow streamlines and its ends are parallel to the shock front. For convenience assume the areas through which the fluid enters and leaves to be unity. The physical laws and definitions listed below will be applied to the flow through the control volume.

- (1) Conservation of mass
- (2) Momentum normal to shock
- (3) Momentum parallel to shock
- (4) Energy equation
- (5) Equation of State
- (6) Geometry of Figure
- (7) 2nd Law of Thermo
- (8) Velocity of sound
- (9) Mach Number

The equations which follow from applications of these nine conditions are

$$\rho_1 N_1 = \rho_2 N_2 \quad (1)$$

$$p_1 + \rho_1 N_1^2 = p_2 + \rho_2 N_2^2 \quad (2)$$

$$\rho_1 N_1 L_1 = \rho_2 N_2 L_2 \quad (3)$$

$$c_p T_1 + \frac{N_1^2 + L_1^2}{2} = c_p T_2 + \frac{N_2^2 + L_2^2}{2} \quad (4)$$



$$\frac{p_1}{\rho_1 T_1} = \frac{p_2}{\rho_2 T_2} \quad (5)$$

$$\tan(\alpha - \omega) = \frac{N_2}{L_2} \quad (6)$$

$$p_{o1} \left( \frac{p_1}{\rho_1^k} \right)^{\frac{1}{k-1}} = p_{o2} \left( \frac{p_2}{\rho_2^k} \right)^{\frac{1}{k-1}} \quad (7)$$

$$\frac{a_1^2}{T_1} = \frac{a_2^2}{T_2} \quad (8)$$

$$\frac{M_1 \sqrt{T_1} \sin \alpha}{N_1} = \frac{M_2 \sqrt{T_2} \sin(\alpha - \omega)}{N_2} \quad (9)$$

The first six equations are sufficient to determine all static conditions across the shock wave for given inlet conditions and  $\omega$  or  $\alpha$ . It will, however, be convenient and useful to solve the complete set of equations simultaneously to obtain the following relations\*

$$\frac{p_2}{p_1} = f_1(M_1 \sin \alpha)$$

$$\frac{T_2}{T_1} = f_3(M_1 \sin \alpha)$$

$$\frac{\rho_2}{\rho_1} = f_2(M_1 \sin \alpha)$$

$$\frac{p_{o2}}{p_{o1}} = f_4(M_1 \sin \alpha)$$

$$M_2^2 \sin^2(\alpha - \omega) = f_5(M_1 \sin \alpha)$$

$$\tan(\alpha - \omega) = \frac{\frac{2}{k+1} + \frac{k-1}{k+1} M_1^2 \sin^2 \alpha}{M_1^2 \sin \alpha \cos \alpha}$$

where the functions  $f_1, f_2$ , etc. are identical with those of note 12.

Solution of the equations proceeds as follows:

Combining (1) and (3) we have

$$R_{N1}(L_1 - L_2) = 0$$

so

$$L_1 = L_2 = L$$

---

\* Some of which are tabulated in "Gas Tables" Tables 55 through 57.



and hence (4) becomes

$$c_p T_1 + \frac{N_1^2}{2} = c_p T_2 + \frac{N_2^2}{2}.$$

Combining (1) with (2) we get

$$p_1 - p_2 = \rho_1 N_1 (N_2 - N_1)$$

or

$$\frac{p_1}{\rho_1 N_1} - \frac{p_2}{\rho_2 N_2} = (N_2 - N_1)$$

which, by substituting  $\frac{a^2}{k}$  for  $\frac{p}{\rho}$  from (8), takes the form

$$\frac{a_1^2}{N_1} - \frac{a_2^2}{N_2} = k(N_2 - N_1). \quad (1)$$

Equations (4) and (8) along with the definition of starred quantities give

$$\left. \begin{aligned} \frac{a^2}{k-1} + \frac{N^2 + L^2}{2} &= a_*^2 \frac{k+1}{2(k-1)} \\ \text{or } a^2 &= \frac{1}{2} \left[ (k+1)a_*^2 - (k-1)(N^2 + L^2) \right] \end{aligned} \right\} \quad (11)$$

Substituting in (1) the values of  $a_2^2$  and  $a_1^2$  found from (11) we find, finally,

$$N_1 N_2 = a_*^2 - L^2 \left( \frac{k-1}{k+1} \right) \quad (111)$$

Notice that for the normal shock  $L = 0$ ,  $N = V$  and (111) reduces to

$$V_1 V_2 = a_*^2.$$

#### Pressure ratio across an oblique shock

Following a procedure analogous to that for the normal shock we get

$$\frac{p_2}{p_1} = f_1(M_1 \sin \alpha).$$

In like manner we find

$$\frac{\rho_2}{\rho_1} = f_2(M_1 \sin \alpha)$$

$$\frac{T_2}{T_1} = f_3(M_1 \sin \alpha)$$

$$\frac{p_{02}}{p_{01}} = f_4(M_1 \sin \alpha)$$

$$M_2^2 \sin^2(\alpha - \omega) = f_5(M_1 \sin \alpha).$$

Expression relating  $M_1$ ,  $\alpha$ , and  $\omega$

If all the shock inlet conditions and  $\alpha$  are given the first four relations above are sufficient to obtain  $p$ ,  $\rho$ ,  $t$ , and  $p$  at the shock exit. The last relation, however, will not give  $M_2$  unless, in addition to  $M_1$  and  $\alpha$ ,  $\omega$  is known. Our next step is to determine  $\omega$  as a function  $M_1$  and  $\alpha$ . By equation we have

$$\tan(\alpha - \omega) = \frac{N_2}{L_2} = \frac{N_1 N_2}{N_1 L_2}$$

using (iii) and expressing velocity components in terms of the resultant velocities we get

$$\tan(\alpha - \omega) = \frac{a_1^2 - \frac{k-1}{k+1} L^2}{V_1^2 \sin \alpha \cos \alpha}$$

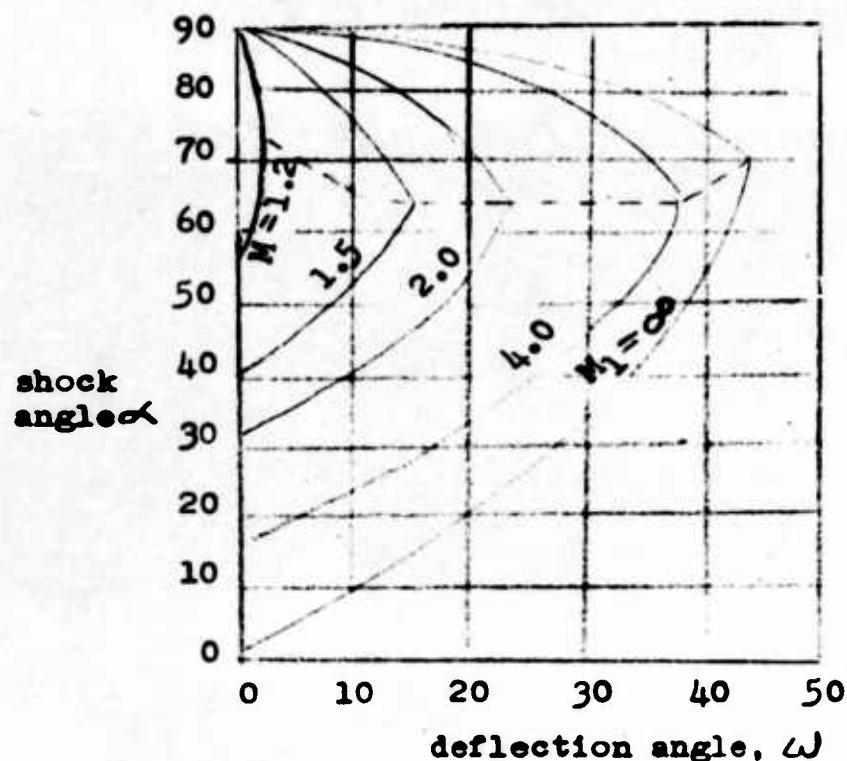
or by equation (ii)

$$\tan(\alpha - \omega) = \frac{\frac{2}{k+1} a_1^2 - \frac{k-1}{k+1} V_1^2 \sin^2 \alpha}{V_1^2 \sin \alpha \cos \alpha}$$

or, dividing numerator and denominator by  $a_1^2$ , we obtain

$$\tan(\alpha - \omega) = \frac{\frac{2}{k+1} + \frac{k-1}{k+1} M_1^2 \sin^2 \alpha}{M_1^2 \sin \alpha \cos \alpha}$$

This expression relates  $M_1$ ,  $\alpha$ , and  $\omega$  and by it we may determine the limitations on  $M_1$  and  $\omega$  for a plane shock to occur when a wedge shaped object is placed in a supersonic flow. It is convenient to present this equation graphically by plotting  $\alpha$  versus  $\omega$  for values of  $M_1$  as shown below.



Observe from the graph that there exists three possible situations for a given wedge angle  $\omega$ . They are

(a) Two values of  $\alpha$  for given  $M_1$ . For example  $\omega = 20^\circ$ ,  $M_1 = 4.0$  give  $\alpha = 32^\circ$  or  $\alpha = 84^\circ$ . Either value of  $\alpha$  may occur depending upon the boundary conditions of the flow. Usually the wave with the larger shock angle occurs. However, with the proper adjustment of the downstream pressure the wave with the lower shock angle may be produced.\*

(b) One value of  $\alpha$  for a given  $M_1$ . For example  $\omega = 23^\circ$ ,  $M_1 = 2.0$ ,  $\alpha = 65^\circ$ .

(c) No value of  $\alpha$  for a given  $M_1$ . For example  $\omega = 20^\circ$ ,  $M_1 = 1.5$ . When this condition exists there occurs in the flow a detached shock wave.

\* See pp. 54-55, Ferri, Elements of Aerodynamics of Supersonic Flows, MacMillan, 1949.

In this analysis there has been developed a series of equations with which it is possible to find the oblique shock exit conditions given the inlet conditions and  $\alpha$  or  $\omega$ . It may be seen that each of the equations in this series reduces to its normal shock counterpart as  $\alpha \rightarrow 90^\circ$  and  $\omega \rightarrow 0^\circ$ . Lastly with the expression relating  $\alpha$ ,  $\omega$ , and  $M_1$ , which is graphed on page 6, we can determine the limiting values of  $M$  and  $\omega$  for an attached shock to occur when a wedge is placed in a supersonic flow. These are the aims we set out to fulfill.

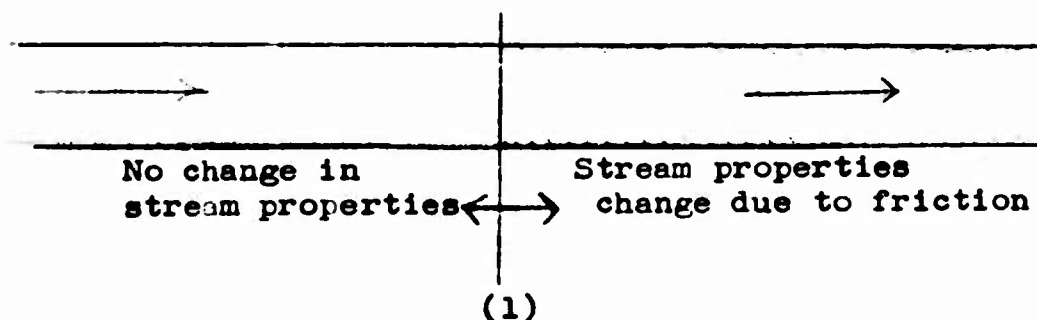
## Lecture Note 14

## SIMPLE FRICTIONAL FLOW - I

The flow to be considered in this note and the next is that of a perfect gas through an adiabatic constant area duct with friction. The purposes of these notes are to determine the locus of the fluid states corresponding to such a flow on the T-s diagram, to discuss the characteristics of this simple frictional flow, and to establish certain expressions that relate the stream properties of the flow to the flow Mach number.

Fanno Line

Consider a perfect gas to be flowing in a frictionless, adiabatic constant area duct. Throughout this flow the stream properties would be invariant. Suppose now there to be joined to this duct at section (1) an adiabatic constant area duct with friction as indicated in the figure. Downstream of (1) the stream properties



will vary due to the presence of friction. A relation between the stream properties at section (1) and the pressure and temperature at any downstream station in the flow may be obtained by writing the energy and continuity equation for the flow. Thus

$$c_p T_1 + \frac{V_1^2}{2} = c_p T + \frac{V^2}{2}$$

$$w = \frac{P}{RT} AV.$$

Replacing  $V$  in the energy equation by  $\frac{w}{A} \frac{RT}{P}$  as obtained from the continuity equation, we get

$$T_1 + \frac{V_1^2}{2c_p} = T + \frac{1}{2c_p} \left( \frac{w}{A} \right)^2 \left( \frac{RT}{P} \right)^2 = T_o \quad (1)$$

For given inlet conditions (therefore given  $T_o$  and  $\frac{w}{A}$ )

this equation represents a relation in terms of temperature and pressure that must be satisfied at any given point in the flow. By assuming values of  $T$  to exist at successive downstream points in the flow it is possible with this relation to determine the corresponding  $p$  at this point thus fixing the state of the fluid ( $p_1, T_1, T_o$ ) at selected points

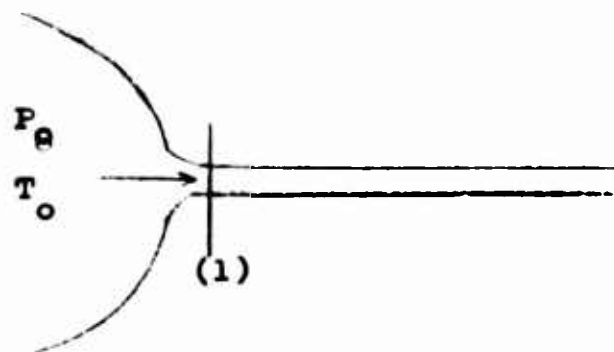
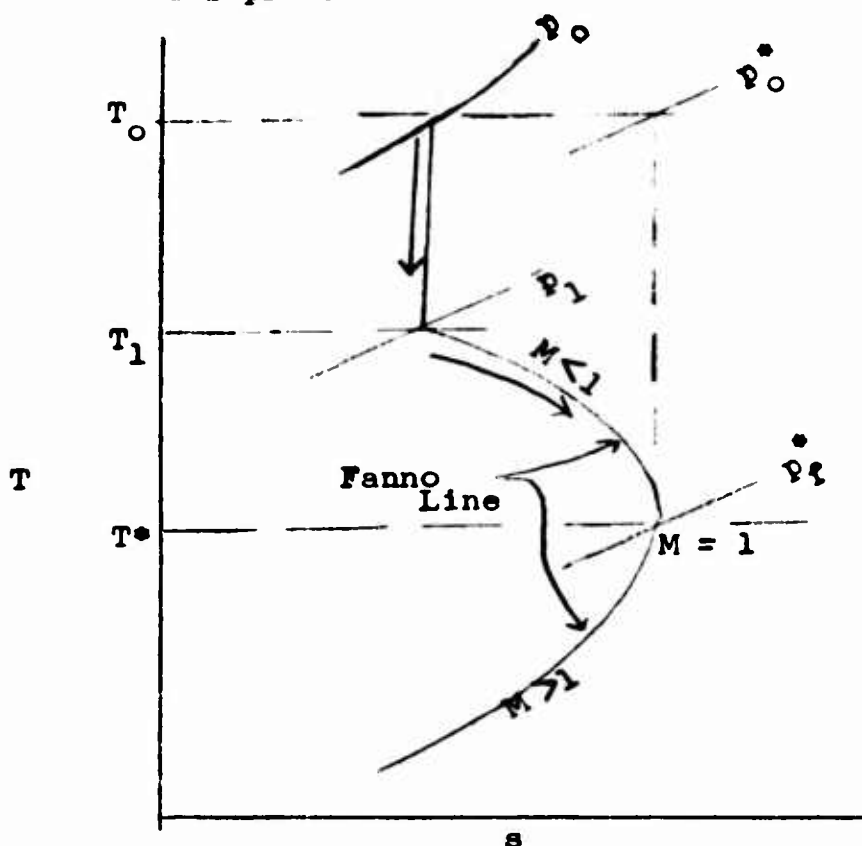
in the flow. Further, by arbitrarily assuming a value of entropy,  $s_1$ , at the inlet to the flow the entropy at downstream points in the flow may be determined by

$$s = \Delta s \Big|_{s_1}^s + s_1$$

$$s = \left[ c_p \ln \frac{T}{T_1} - R \ln \frac{p}{p_1} \right] + s_1 \quad (ii)$$

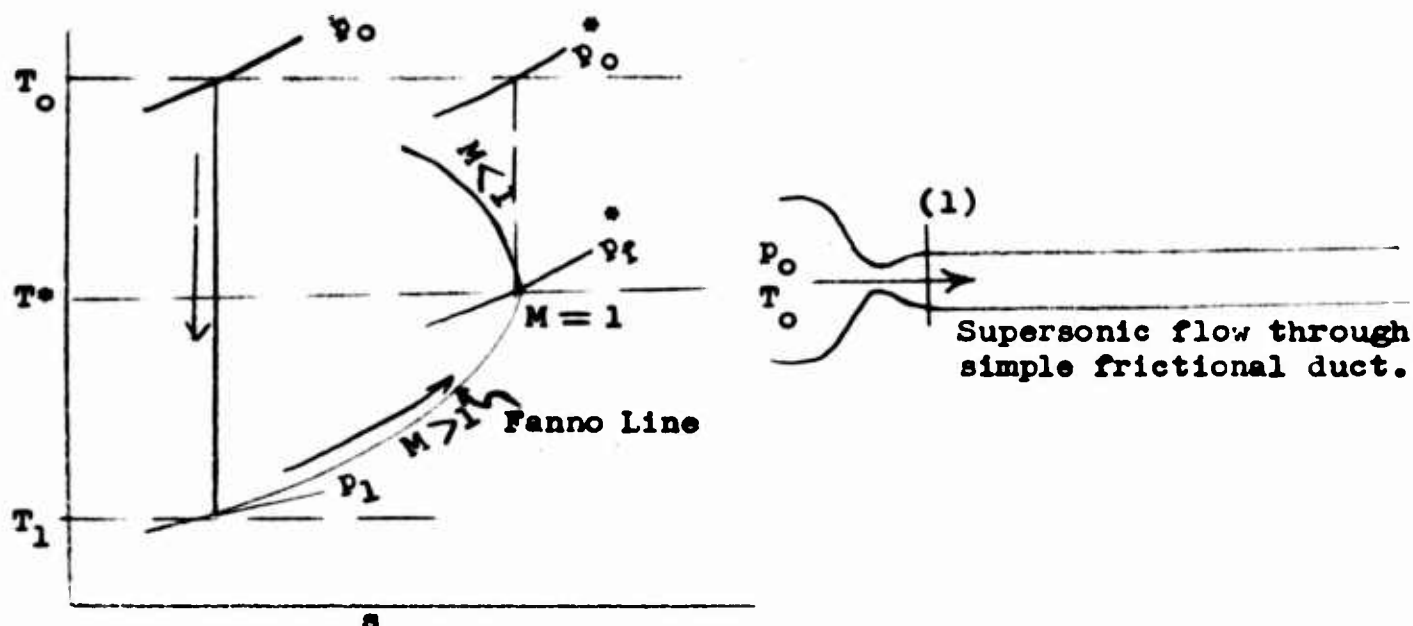
The Second Law through equation (ii) further restricts the values of  $T$  and  $p$  that may exist downstream of the inlet by the fact that the fluid must proceed through values of  $T$  and  $p$  corresponding to states of increasing entropy since the flow process is an irreversible adiabatic process.

As an illustration consider a perfect gas to be flowing from a large reservoir through a convergent nozzle thence through a simple frictional duct. With known inlet conditions at section (1) (figure below) we may, by assuming values of  $T$  to exist downstream as a result of the frictional effects in the flow, determine with equations (i) and (ii) the locus of states the fluid may attain in the flow. This locus plotted on a  $T$ - $s$  diagram is called a Fanno Line and appears as indicated on the accompanying  $T$ - $s$  plot.



Subsonic flow through simple frictional duct.

Beginning at state  $(p_o, T_o)$  on the  $T-s$  diagram we find the flow to proceed isentropically to  $(p_1, T_1)$  thence along the Fanno Line through states of increasing entropy and increasing Mach number tending toward a Mach number of 1. For the flow to proceed beyond  $M = 1$  would require a decrease of entropy in violation of the Second Law. Thus we find in an initially subsonic simple frictional flow that the Mach number increases toward a limiting value of one. Similarly, in an initially supersonic simple frictional flow (figure below) the Mach number follows along the lower branch of a Fanno Line through states of higher entropy and lower Mach numbers toward a limiting Mach number of one. It is impossible, therefore, for a flow to proceed along a Fanno Line or through a simple frictional duct continuously from subsonic to supersonic or from supersonic to subsonic conditions.



For given inlet conditions to a simple frictional duct, there exists a Fanno Line representing the possible states that the flow may proceed through in the duct. Whether a portion or all of these possible states are attained by the fluid as it flows through the duct depends upon the amount of frictional duct length and the pressures imposed upon the boundaries (inlet and exit) of the flow system. Let us examine the effects of frictional duct length and "boundary pressures" upon a given system. Suppose, for example, that we have a convergent nozzle - simple frictional duct unit and consider in turn the effects of

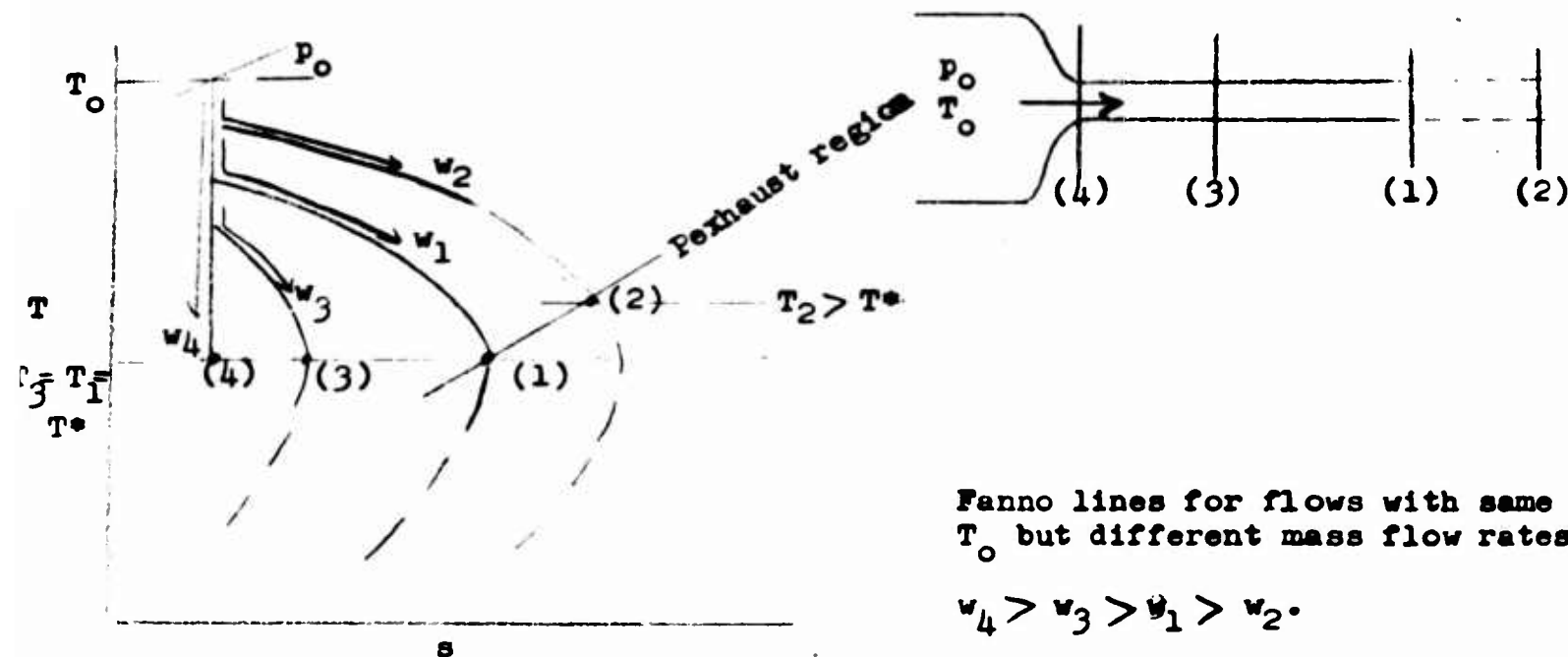
(a) frictional duct length

(b) reservoir pressure

on the flow through the unit.

### Effect of Frictional Duct Length in Subsonic Flow.

To examine (a) we will assume constant reservoir and exhaust region pressures and let the duct length vary. As a starting point let the unit be such that at its exit  $M = 1$  and the exhaust region pressure is just attained.



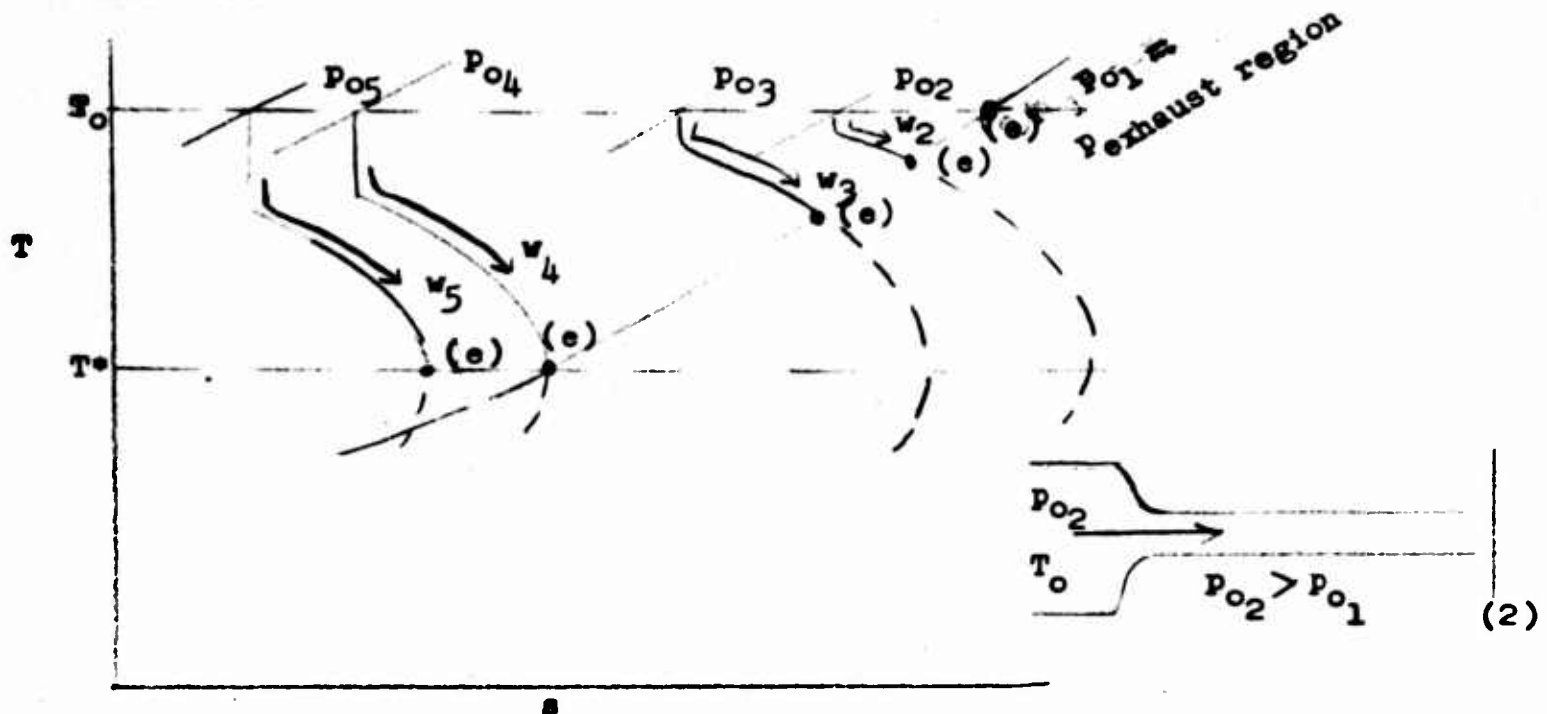
For this case (figure above) the flow through the unit from  $(p_0, T_0)$  follows isentropically down to the inlet of the simple frictional duct thence along a Fanno Line corresponding to  $w_1$  to  $M = 1$  at duct exit section (1). If now the duct length is increased to (2), everything else remaining the same, we find the flow process to follow along a Fanno Line corresponding to a lower mass rate of flow  $w_2$ . Throughout this latter flow  $M < 1$ . If the duct length is increased further beyond (2) the mass flow in the unit continues to decrease and in the limit  $w$  tends to zero as the duct length tends to infinity. Now, on the other hand, if the duct length is decreased to (3), we find the flow process to proceed isentropically down to the duct inlet and thence along a Fanno Line of mass rate of flow  $w_3 > w_1$ . As the duct length goes to zero, the Mach number of the nozzle throat increases to a Mach number of one corresponding to a maximum mass rate of flow  $w_4$  through the nozzle (as the duct length goes to zero the unit becomes a simple convergent nozzle).

### Effect of Reservoir Pressure in Subsonic Flow.

Consider next (b) that is the effect of reservoir pressure on a given unit. First let the reservoir pressure,  $p_{01}$ , and exhaust region pressure be the same.



For this case there is no flow. If now the reservoir pressure is increased\* to a value  $P_{o2}$  slightly above the exhaust region pressure flow will start with a low mass rate  $w_2$  through the unit  $M < 1$  everywhere. The process is indicated



on the T-s diagram as a vertical line from  $P_{o1}, T_o$  to the inlet of the simple frictional duct and thence along a Fanno line<sup>2</sup> corresponding to a mass flow  $w_2$  to the exhaust pressure at the exit section (e) of the duct. As the total pressure increases the mass flow increases and  $w_5 > w_4 > w_3 > w_2 > w_1 = 0$ . The exit Mach number of the unit remains constant at unity for  $P_o > P_{o4}$  (as indicated in the figure). The expansion from  $P_e$  to  $P_{\text{exhaust region}}$  for  $P_o > P_{o4}$  takes place in the exhaust region.

It has been assumed in the above that the simple frictional duct was preceded by a convergent nozzle and hence experienced only inlet conditions corresponding to subsonic flow. Let us next examine (a) and (b) as noted above (effect of duct length and reservoir pressure on flow) for the case in which the simple frictional duct is attached to a convergent-diverging nozzle which may provide supersonic inlet conditions to the frictional duct. We find now that in order for some values of duct length and reservoir and exhaust region pressures applied to the flow to be satisfied, discontinuities in the form of normal shock waves must exist in the flow. This situation

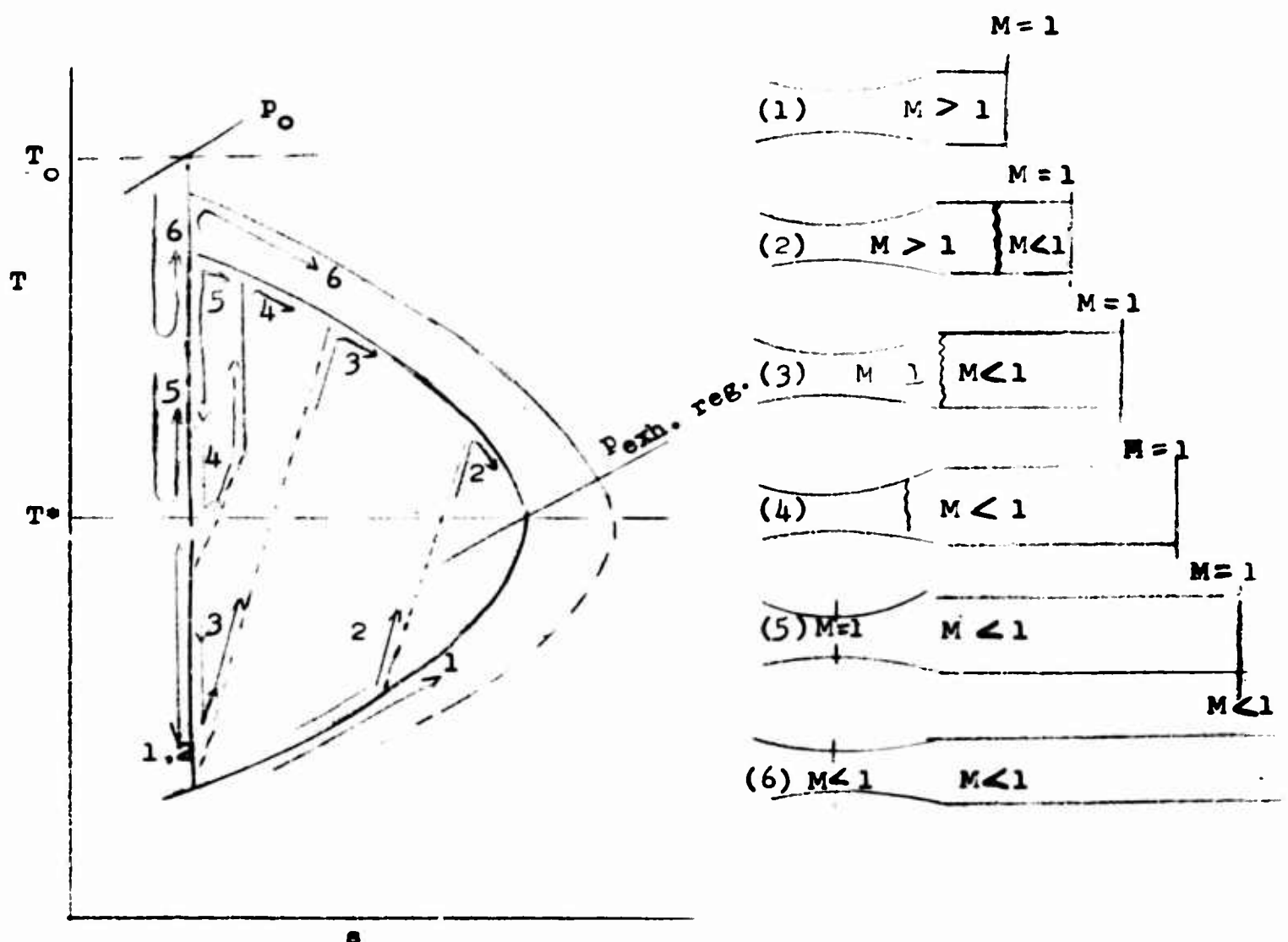
\* To increase the reservoir pressure at constant temperature will require cooling of the reservoir. Let the reservoir pressure be increased reversibly and isothermally then the change in reservoir entropy is given by

$$\Delta s = \left( \frac{\partial s}{\partial T} \right)_{\text{rev}} \quad \text{and since } q < 0 \text{ (cooling) then } \Delta s < 0.$$

is analogous to that of a frictionless nozzle operating with a normal shock in its divergent section.

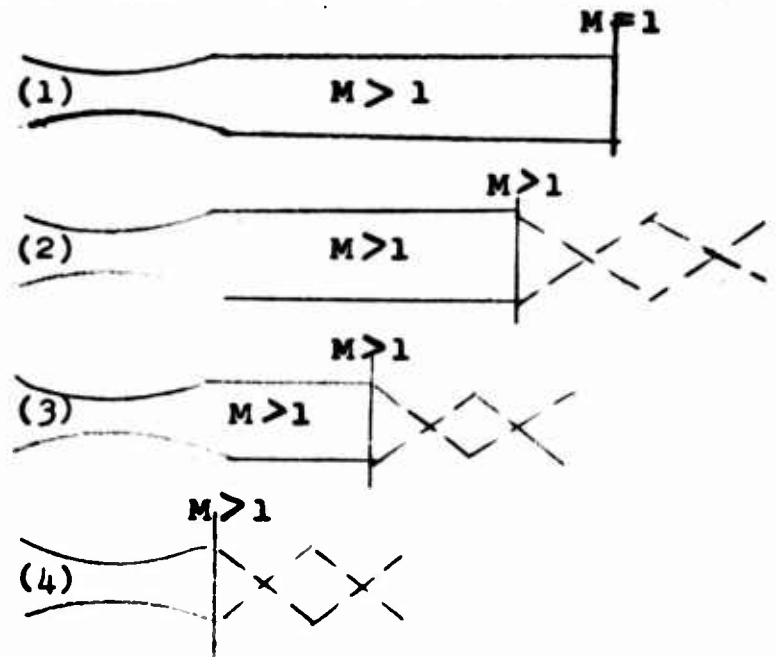
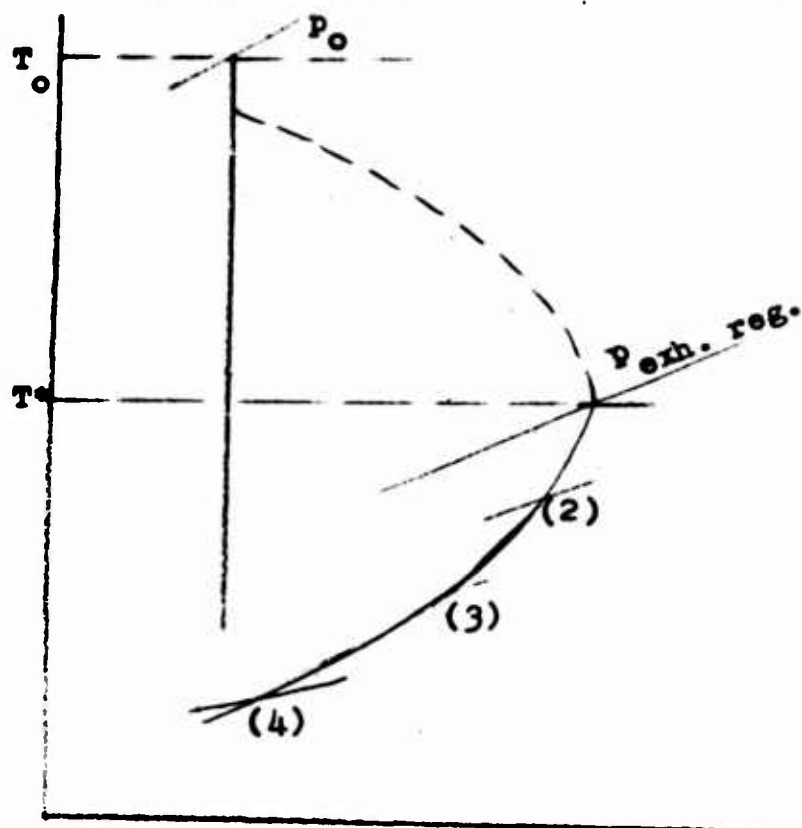
### Effect of Frictional Duct Length in Supersonic Flow.

As an example, assume a supersonic nozzle - simple frictional duct unit and consider the effect of (a), frictional duct length, on the flow through the unit. Initially let the unit be operating such that the flow leaving the unit is at  $M = 1$  and at exhaust region pressure with a mass rate of flow  $w_1$  through it. This condition is indicated on the  $T-s$  diagram below where the flow process originates at  $p_0, T_0$  and proceeds isentropically to the supersonic branch of the Fanno line corresponding to  $w_1$  and then follows along this Fanno line to  $M = 1$  and  $p_{\text{exhaust}}$  region. Now as the duct length is increased to (2) we find that the new boundary condition of increased duct length can be satisfied by assuming a normal shock to occur at a point in the duct such that the combination of duct length preceding and following the flow discontinuity produce a Mach number of one at the duct exit. The flow process corresponding to this condition is shown by the arrows numbered 2.



As the duct length is increased further the normal shock progresses upstream to the duct inlet, thence into the nozzle until it reaches the nozzle throat (5). Further increase in length beyond that corresponding to (5) reduces the mass rate of flow through the unit and the flow progresses through the duct for these cases along Fanno lines of lower mass flows as indicated by (6).

Suppose now a flow corresponding to condition (1) above exists and let the duct length be reduced. In this case we find that the stream properties in the remaining portion are unaffected and as the duct length is reduced to zero the flow reduces to that through a convergent-divergent nozzle exhausting to the discharge region through a system of oblique shock waves set up in the exhaust region. These conditions are illustrated schematically below. For (1) the flow proceeds isentropically to the duct inlet, thence along a Fanno



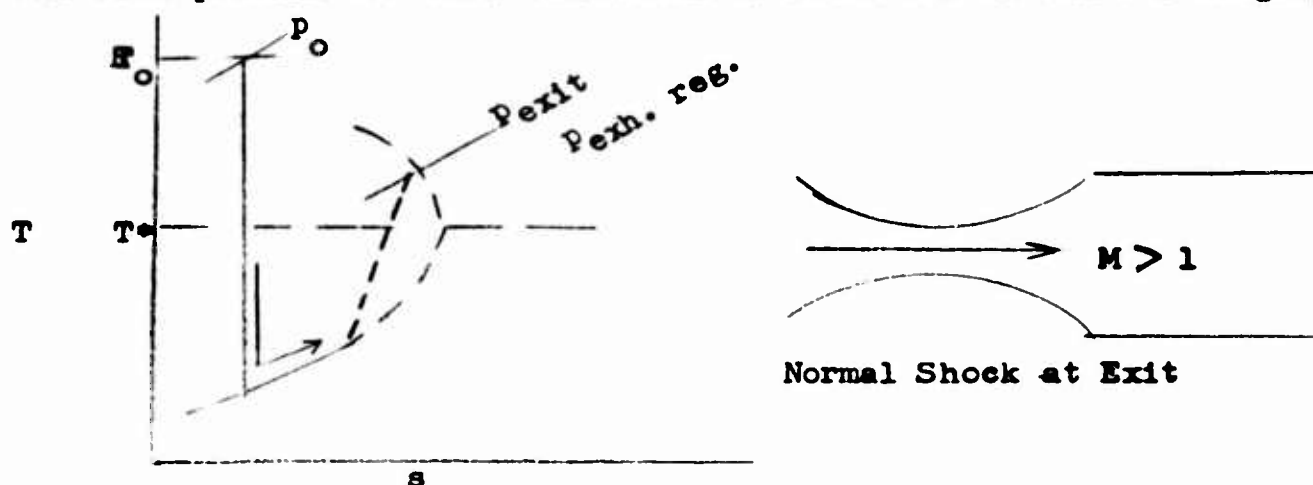
Oblique shock waves present in discharge region in cases (2), (3), and (4).

line to  $M = 1$  at (1). For (2), (3) and (4) the exit conditions from the unit are as indicated on the T-s diagram. Notice that  $p_2$ ,  $p_3$  or  $p_4$  are each less than  $p_{\text{exhaust region}}$ . The rise in pressure to  $p_{\text{exhaust}}$  in these cases is attained through a series of oblique shock waves set up from the exit of the duct.

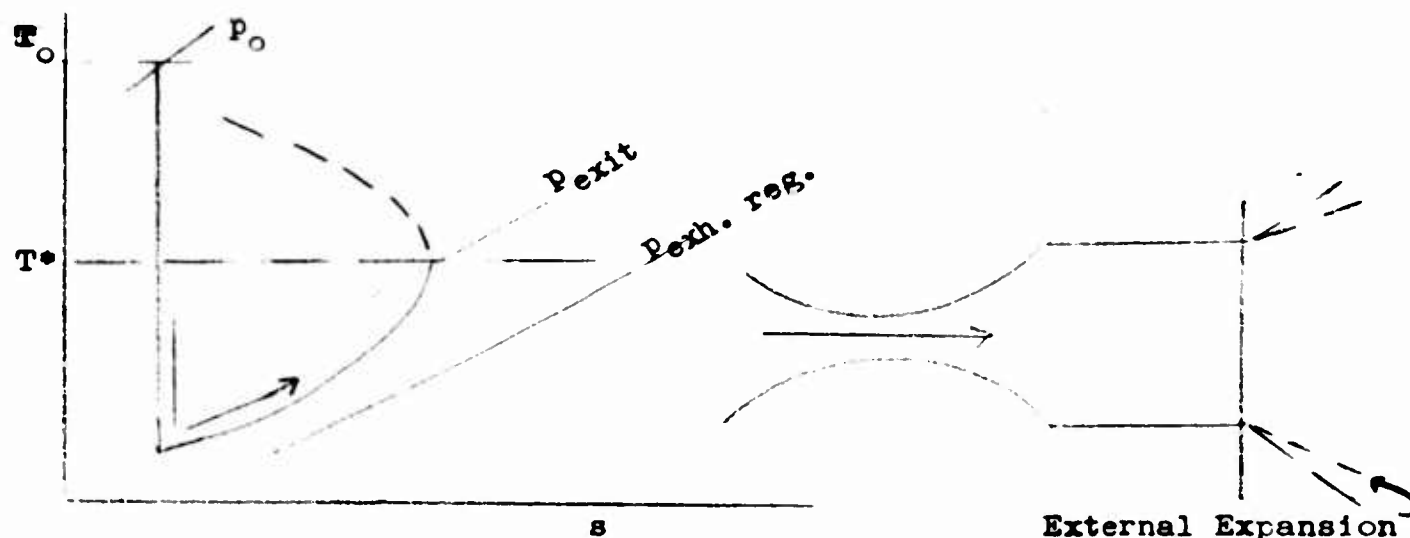
#### Effect of Reservoir Pressure in Supersonic Flow.

As the last consideration of this note, let us examine the effect of reservoir source pressure on supersonic simple frictional flow. Starting with

the reservoir pressure of our convergent-diverging nozzle-simple frictional duct unit equal to the exhaust pressure, we have no flow. Now as the reservoir pressure increases the flow through the unit increases. Finally a Mach of one is reached in the nozzle throat and at the duct exit. Up to this point the flow processes for each reservoir  $p_0$  appear on a T-s diagram as for condition (6), page 14.6. For reservoir pressures beyond that just giving sonic throat conditions, a normal shock arises downstream of the nozzle throat and progresses downstream as the reservoir pressure and hence mass flow are increased. A typical flow process for this condition is that of (4) page 14.6. As the reservoir pressure is increased further the normal shock reaches the duct exit. The flow process for this condition is indicated on the T-s diagram below.



For reservoir pressures above that producing a normal shock at the duct exit the flow process appears as for condition (3) page 14.7 with oblique shocks at the exit of the duct. These oblique shocks become weaker and ideally disappear as the total pressure reaches a value producing a duct exit pressure corresponding to the discharge region pressure. This condition corresponds to that of condition (1) page 14.7. Lastly, with the reservoir pressure increased further we find the mass flow to increase and the flow Mach number to remain constant, such that  $M = 1$  at duct exit. This condition is illustrated in the sketch below. The transition from the higher duct exit pressure to the discharge region pressure



takes place in the exhaust region. The various flow processes obtained as the reservoir pressure is increased from that corresponding to the exhaust region pressure to that giving the condition indicated in the sketch above are shown on a single T-s diagram on the following page.

In this note the Fanno line has been presented and its use illustrated. The effect of

(a) frictional duct length

and

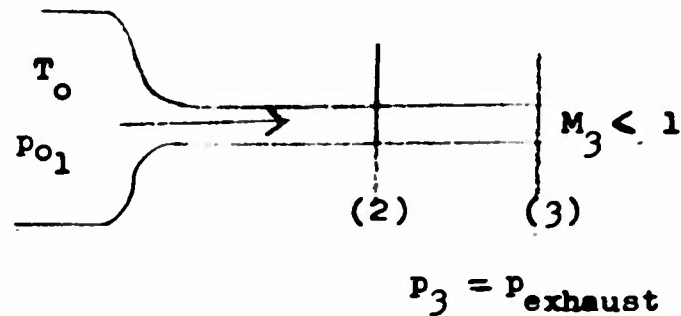
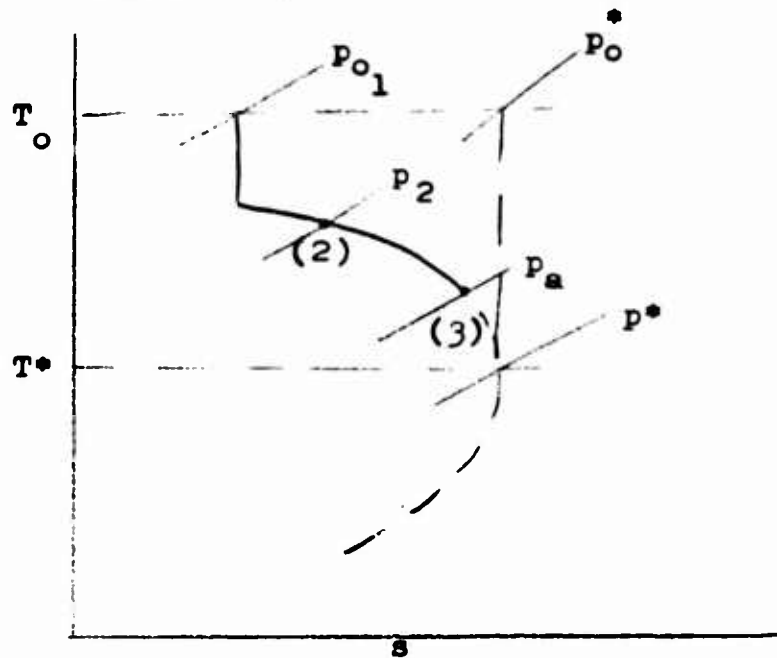
(b) reservoir pressure

on the stream flow properties of a simple frictional duct attached in turn to a convergent nozzle and a supersonic nozzle have been described. In the succeeding note the simple frictional flow will be investigated analytically and relations between the flow stream properties and the stream Mach numbers will be obtained. These relations will permit a simple quantitative analysis of the flow under consideration.

## Lecture Note 15

## SIMPLE FRICTIONAL FLOW - II

An analysis identical in method to that of note 10 only applied to simple frictional flow will be made in this note. It will be shown that a stream property at any given station in a simple frictional flow divided by its value corresponding to the point where  $M = 1$  on the Fanno line of the flow is a function of the Mach number at the given station. Thus, for example, the properties of the stream at section 2 of the flow unit depicted below divided by the starred



values of these properties give

$$\left(\frac{P}{P^*}\right)_2 = f_1(M_2)$$

$$\left(\frac{P}{P_o^*}\right)_2 = f_3(M_2)$$

$$\left(\frac{P_o}{P_o^*}\right)_2 = f_2(M_2)$$

$$\left(\frac{T}{T^*}\right)_2 = f_4(M_2) \text{ etc.}$$

Further it will be shown that the duct length beyond station (2) required to cause the flow to attain a Mach number of 1 at the duct exit is a function of Mach number at section (2). This length will be called  $L_{\max_2}$  and we will find

that this length multiplied by the constant  $4f/D$  where  $f$  is duct friction factor and  $D$  the duct diameter gives a relation such that

$$\left(\frac{4f}{D} L_{\max}\right)_2 = f(M_2).$$

To obtain the above relations we write equations involving the stream properties of the flow in terms of logarithmic differentials. Towards this end let us first apply the momentum equation to the flow under study and reduce it to a form involving logarithmic differentials. We have

$$F = F_1 - F_2$$

and between any two sections separated an infinitesimal distance  $dx$

$$\delta F = -dF$$

$$\text{Now } F = pA + \rho A V^2 \quad = pA + wV$$

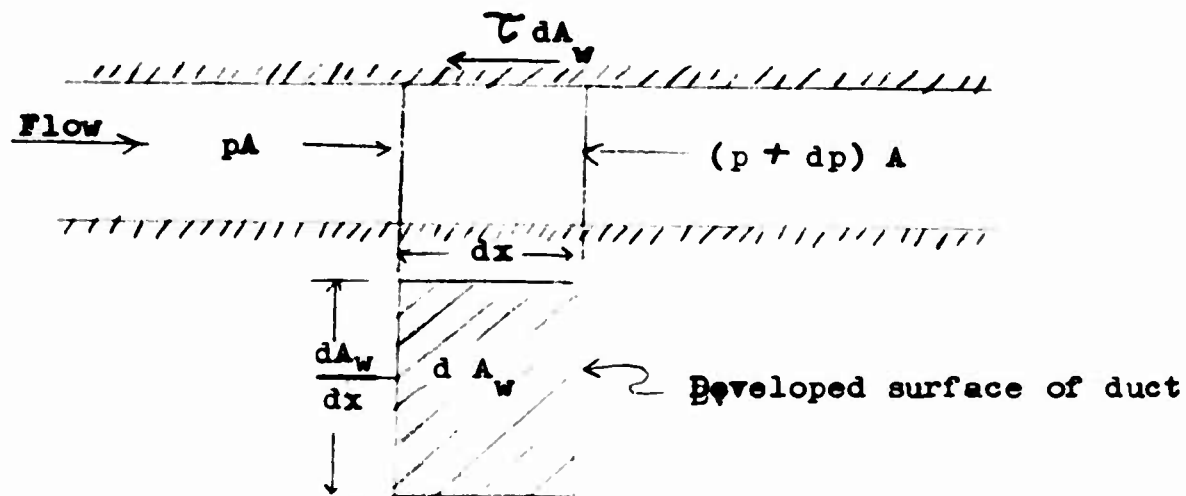
$$\text{so } dF = A dp + w dV$$

and  $\delta F$  for case of  $dA = 0$  is due only to frictional forces and is given by (figure below)

$$\delta F = \tau dA_w$$

where  $\tau$  = shearing force of duct on fluid per unit of duct wetted area.

$A_w$  = wetted duct area.



$\delta F$  is used here to indicate an infinitesimal force, of fluid on duct, and is not an exact differential as compared to  $dF$ .

There follows after substituting into equation (i)

$$\tau dA_w = -A dp - w dV. \quad (11)$$

Introducing the duct-fluid coefficient of friction defined as

$$f = \frac{\tau}{\frac{\rho V^2}{2}}$$

and the hydraulic diameter D defined as

$$D = 4 \frac{\text{x-sectional area}}{\text{wetted perimeter}}$$

$$D = 4 \frac{A}{dA_w/dx} = 4A \frac{dx}{dA_w}.$$

we obtain

$$\tau dA_w = f \frac{\rho V^2}{2} \frac{4A dx}{D}.$$

Placing this result in (ii) the momentum equation takes the form

$$A dp + w dV + f \frac{\rho V^2}{2} \frac{4A}{D} dx = 0.$$

Dividing this equation by Ap we obtain

$$\frac{dp}{p} + \frac{\rho V^2}{p} \frac{dV}{V} + \frac{\rho V^2}{p} \frac{1}{2} \left( \frac{4f dx}{D} \right) = 0$$

where

$$\frac{\rho V^2}{p} = \frac{V^2}{RT} = \frac{kV^2}{kRT} = kM^2$$

so finally

$$\frac{dp}{p} + kM^2 \frac{dV}{V} + \frac{kM^2}{2} \left( \frac{4f dx}{D} \right) = 0 \quad (111)$$

This equation along with those obtained from the equation of state etc. give the



following system of equations to be satisfied simultaneously in the simple frictional flow of a fluid,

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \quad (\text{state})$$

$$\frac{dT}{T} + \frac{\frac{k-1}{2} M^2}{1 + \frac{k-1}{2} M^2} \frac{dM^2}{M^2} = 0 \quad (\text{energy})$$

$$\frac{d\rho}{\rho} + \frac{dV}{V} = 0 \quad (\text{continuity})$$

$$\frac{dp}{p} + \frac{kM^2}{1 + \frac{k-1}{2} M^2} \frac{dM^2}{M^2} = \frac{dp_0}{p_0} \quad (\text{total pressure})$$

$$\frac{dp}{p} + kM^2 \frac{dV}{V} + \frac{kM^2}{2} \left( \frac{4f dx}{D} \right) = 0 \quad (\text{momentum})$$

$$\frac{dM^2}{M^2} = 2 \frac{dV}{V} - \frac{dT}{T} \quad (\text{Mach number})$$

$$\frac{dF}{F} = \frac{dp}{p} + \frac{kM^2}{1 + kM^2} \frac{dM^2}{M^2} \quad (\text{impulse function})$$

$$\frac{ds}{c_p} = \frac{dT}{T} - \frac{k-1}{k} \frac{dp}{p} \quad (\text{entropy})$$

This constitutes a system of eight equations in nine variables -  $p$ ,  $\rho$ ,  $T$ ,  $M$ ,  $V$ ,  $p_0$ ,  $x$ ,  $f$ ,  $s$ . We select one as the independent variable and solve for each of the remaining dependent variables in terms of the independent variable. Selecting, therefore,  $\frac{4f dx}{D}$  as the independent variable these equations may be combined to give each dependent variable as a function of  $\frac{4f dx}{D}$ . As an example consider obtaining the relationship

$$\frac{dM^2}{M^2} = f \left( \frac{4f dx}{D} \right)$$

One procedure is outlined herewith.

- (a) Using the state equation obtain  $p, \rho, T$  related
- (b) Using the momentum equation obtain  $V, x, \rho, T$  related
- (c) Using the continuity equation obtain  $V, x, T$  related
- (d) Using the Mach equation obtain  $M^2, x, T$  related
- (e) Using the energy equation obtain  $M^2, x$  related.

The result is

$$\frac{dM^2}{M^2} = \frac{kM^2 \left(1 + \frac{k-1}{2} M^2\right)}{1 - M^2} \frac{4f dx}{D}.$$

There are summarized below the relations between all independent variables and  $\frac{4f dx}{D}$ .

	$4f dx/D$
$\frac{dM^2}{M^2}$	$\frac{kM^2 \left(1 + \frac{k-1}{2} M^2\right)}{1 - M^2}$
$\frac{dV}{V}$	$\frac{kM^2}{2(1 - M^2)}$
$\frac{dp}{p}$	$-\frac{kM^2 [1 + (k-1)M^2]}{2(1 - M^2)}$
$\frac{d\rho}{\rho}$	$-\frac{kM^2}{2(1 - M^2)}$
$\frac{dT}{T}$	$-\frac{k(k-1)M^4}{2(1 - M^2)}$
$\frac{dp_0}{p_0}$	$-\frac{kM^2}{2}$
$\frac{dF}{F}$	$-\frac{kM^2}{2(1 + kM^2)}$
$\frac{ds}{c_p}$	$\frac{(k-1)M^2}{2}$

The above results indicate that the sign of variation of each of the stream properties  $M$ ,  $V$ ,  $p$ ,  $\rho$ , and  $T$  with duct length  $x$  depends only upon the flow Mach number. Thus, for example,

$$\frac{dV}{dx} = \frac{kM^2}{2(1 - M^2)} \frac{4f}{D} V$$

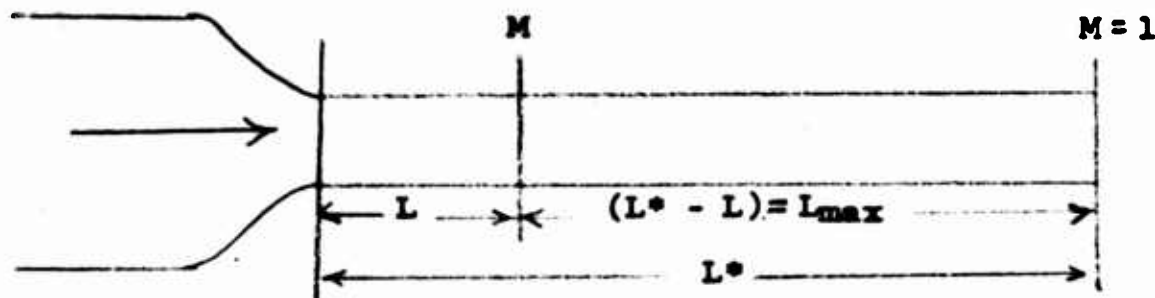
and friction accelerates the flow velocity in subsonic flow and decelerates the flow in a supersonic stream.

$4fL_{\max}/$  and the ratio  $p/p^*$ ,  $T/T^*$  etc.

It is possible to determine the pipe length required beyond any given station in a frictional flow to give  $M = 1$  in the flow by integrating the relation

$$\frac{4f dx}{D} = f(M^2) \frac{dM^2}{M^2}$$

from any pipe station  $L$  and Mach number  $M$  to the pipe station  $L^*$  where Mach number is unity.



During the integration we assume  $f$  to be constant. Now

$$\frac{4f}{D} \int_L^{L^*} dx = \int_{M^2}^1 \frac{1 - M^2}{kM^2(1 + \frac{k-1}{2} M^2)} \frac{dM^2}{M^2}$$

\*\* To obtain  $M = 1$  without affecting the initial values of the stream properties in a pipe would require proper adjustment in exhaust region pressure as the pipe length is increased.

Let  $y = M^2$

$$a = \frac{k-1}{2}$$

then

$$\frac{4f}{D}(L^* - L) = \int_y^1 \frac{(1-y)}{ky(1+ay)} \frac{dy}{y} = \frac{k}{k} \int_y^1 \frac{dy}{y^2(1+ay)} - \frac{1}{k} \int_y^1 \frac{dy}{y(1+ay)}$$

Using partial fractions we find

$$\int_y^1 \frac{dy}{y^2(1+ay)} = \frac{1-y}{y} - \ln \left[ \frac{1+ay}{(1+a)y} \right]$$

and

$$\int_y^1 \frac{dy}{y(1+ay)} = \ln \left[ \frac{1+ay}{(1+a)y} \right]$$

There is obtained finally

$$\frac{4fL_{\max}}{D} = \frac{1-M^2}{kM^2} + \frac{k+1}{2k} \ln \left[ \frac{(k+1)M^2}{2(1+\frac{k-1}{2}M^2)} \right]$$

This relation is plotted in the accompanying sketch. This graph shows that the effect of friction on the stream properties is much greater in supersonic flow than in subsonic flow. For a pipe of 1" diameter with an  $f$  of 0.01 the

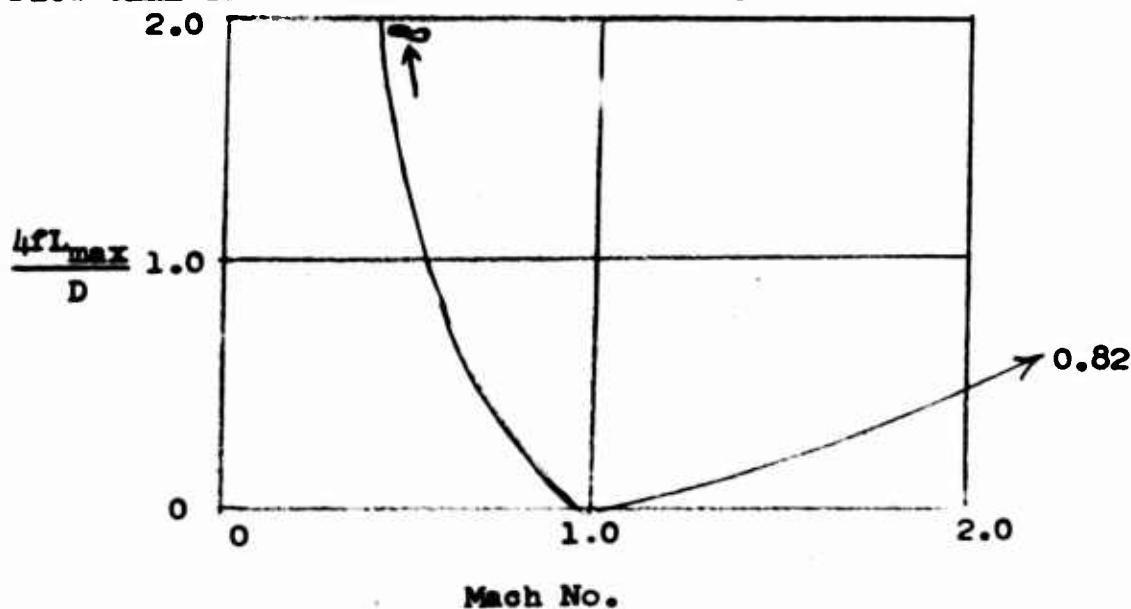
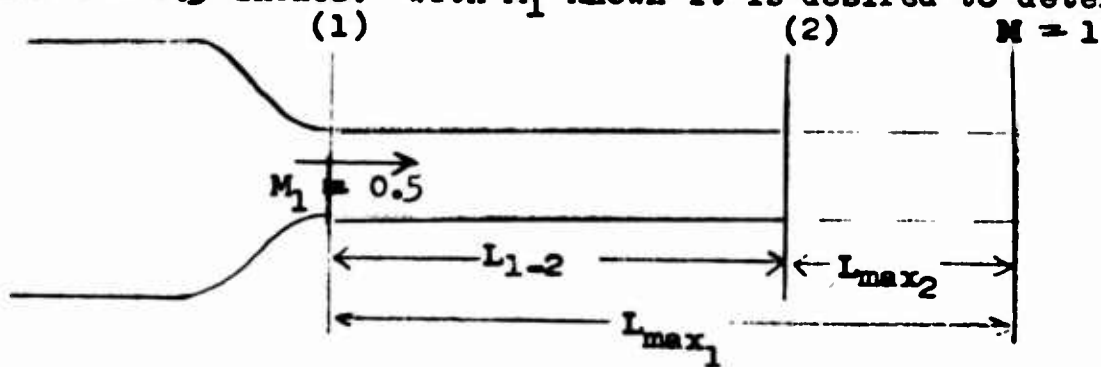


figure indicates that to cause a supersonic flow to reduce from  $M = \infty$  to  $M = 1$  requires  $\frac{4fL_{\max}}{D} = 0.82$  or a length of  $L_{\max} = \frac{0.82 \times 1}{4 \times 0.01} = 20$  inches.

Whereas in subsonic flow the effect of friction is such that a 20 inches pipe length is required to change the flow Mach number from 0.65 to 1.0.

To illustrate the use of the above relation consider the following example. Assume a flow exists as shown below with  $f = 0.0025$  and pipe diameter  $D = 0.5$  inches. With  $M_1$  known it is desired to determine the exit



Mach number from the pipe of length 50 inches.

Solution:

$$M_1 = 0.5 \text{ gives } \left( \frac{4fL_{\max}}{D} \right)_1 = 1.07 \text{ (Table 42)}$$

$$L_{\max 2} = L_{\max 1} - L_{1-2}$$

so, multiplying through by  $4f/D$ ,

$$\left( \frac{4fL_{\max}}{D} \right)_2 = \left( \frac{4fL_{\max}}{D} \right)_1 - \left( \frac{4fL_{1-2}}{D} \right)$$

$$= 1.07 - \frac{4 \times 0.0025}{0.5} \times 50 = 1.07 - 1$$

$$\left( \frac{4fL_{\max}}{D} \right)_2 = 0.07$$

## The interpretation of the relation

$$\frac{4f_{l_{\max}}}{D} = r(M)$$

$$\frac{dp}{p} = \rho_1(M^2) \frac{dM^2}{M^2} \quad \frac{dp_0}{p_0} = \rho_3(M^2) \frac{dM^2}{M^2}$$

$$\frac{dT}{T} = \rho_2(M^2) \frac{dM^2}{M^2} \quad \frac{dT}{T} = \rho_4(M^2) \frac{dM^2}{M^2} \quad \text{etc.}$$

Each of which may be integrated between  $(M^2, p)$  and  $M^2 = 1, p = p^*$  etc. to give

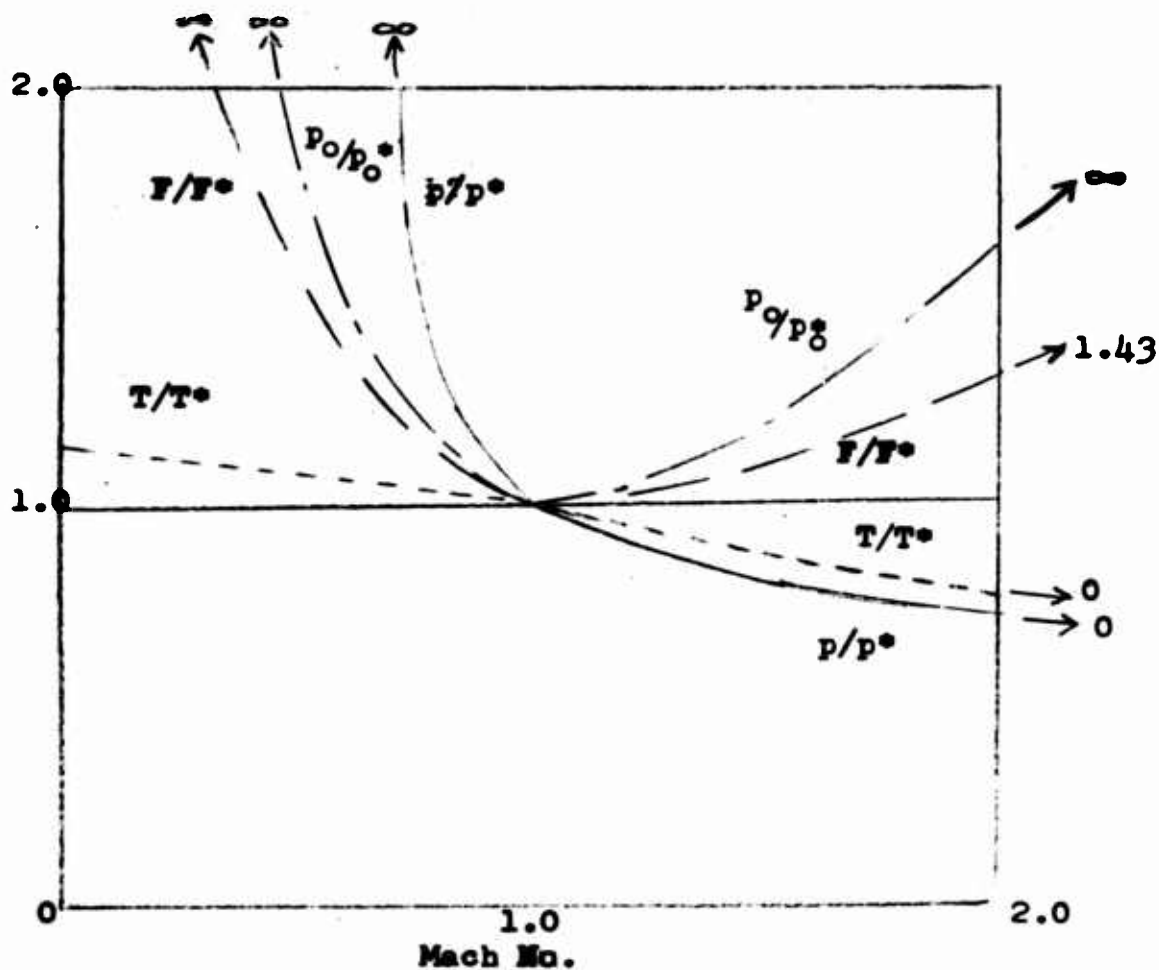
$$\frac{p}{p^*} = f_1(M)$$

$$\frac{p_0}{p_0^*} = f_3(M)$$

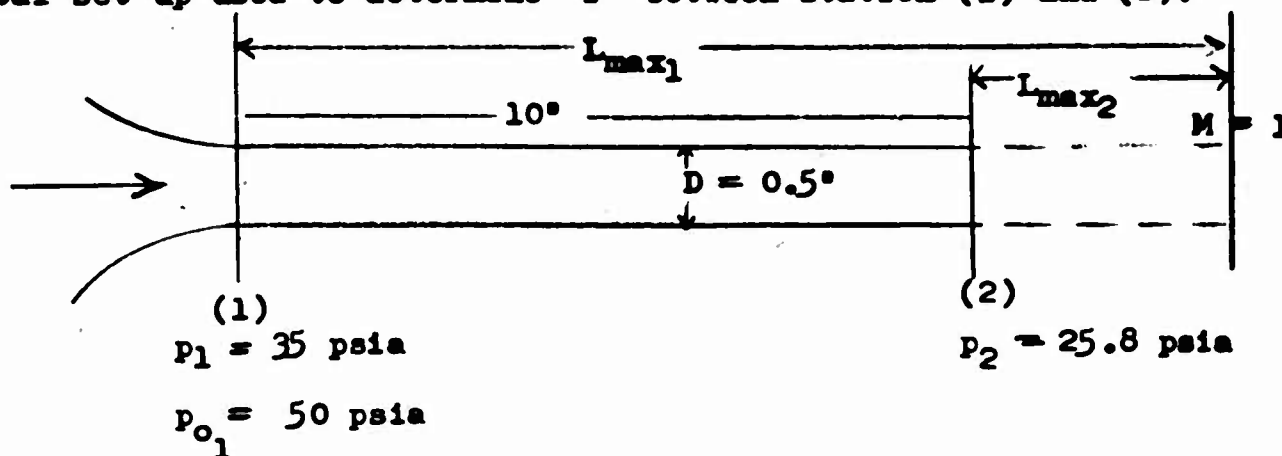
$$\frac{T}{T^*} = f_2(M)$$

$$\frac{F}{F^*} = f_4(M) \text{ etc.}$$

Thus we find, for example, that the total pressure at any station (a) in simple frictional flow divided by the total pressure corresponding to  $M = 1$  in the flow depends only upon the Mach number at (a). The ratios  $p/p^*$  etc. are plotted versus Mach number below and are tabulated in "Gas Tables" by Keenan and Kaye.



As an example of the use of these tables consider the following experimental set up used to determine  $f$  between station (1) and (2).



Solution:  $p_{01} = 50 \text{ psia}$

$$\frac{p}{p_{01}} = 0.7 \text{ so } M_1 = 0.73 (\text{Isentropic Table})$$

With  $M_1 = 0.73$ ;  $\frac{p}{p^*} = 1.426$  (Fanno Table)

$$\text{Now } \left( \frac{p}{p^*} \right)_2 \left( \frac{p}{p^*} \right)_1 \frac{p_2}{p_1} = 1.426 \frac{25.8}{35} = 1.05$$

Whence  $M_2 = 0.955$  (Fanno Table)

With  $M_1$  and  $\left( \frac{p}{p^*} \right)_2$  known we find

$$\frac{4fL_{\max}}{D} = 0.156; \quad \frac{4fL_{\max}}{D} = 0.0026$$

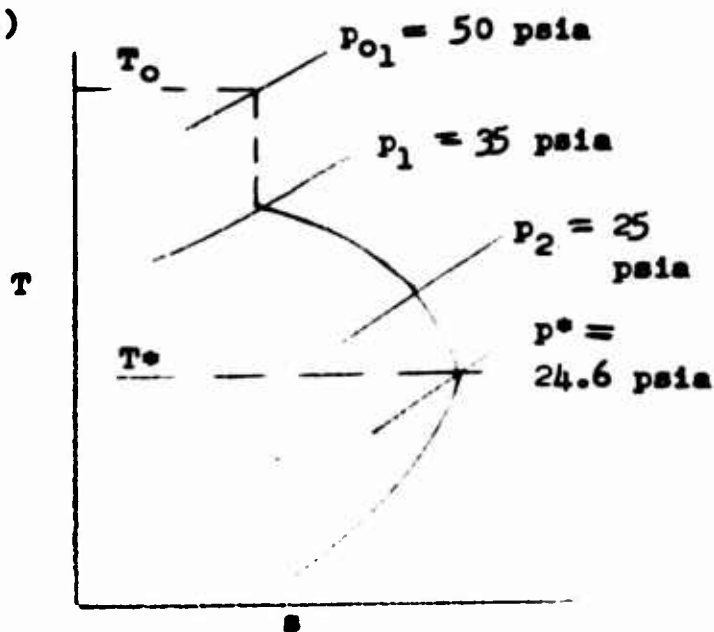
$$L_{\max 1} = L_{1-2} + L_{\max 2}$$

or multiplying by  $4f/D$  we have

$$\left( \frac{4fL_{\max}}{D} \right)_1 = \frac{4fL_{1-2}}{D} + \left( \frac{4fL_{\max}}{D} \right)_2$$

$$\text{giving } \frac{4fL_{1-2}}{D} = 0.156 - 0.0026$$

$$\text{from which we find } f = \frac{(0.153)(0.5)}{4 \times 10} = 0.0019.$$





## Lecture Note 16

SIMPLE  $T_0$  FLOW - I

Thus far we have considered the steady one-dimensional flow of a perfect gas with simple area change and simple frictional effects respectively. We analyze next the flow of a perfect gas with simple heating effects. This type of flow may alternatively be called simple  $T_0$  flow since we treat as our independent variable the total temperature  $T_0$ . This is controlled through heating as noted in the steady flow energy equation with no shaft work

$$\frac{Q}{c_p} = T_{02} - T_{01}$$

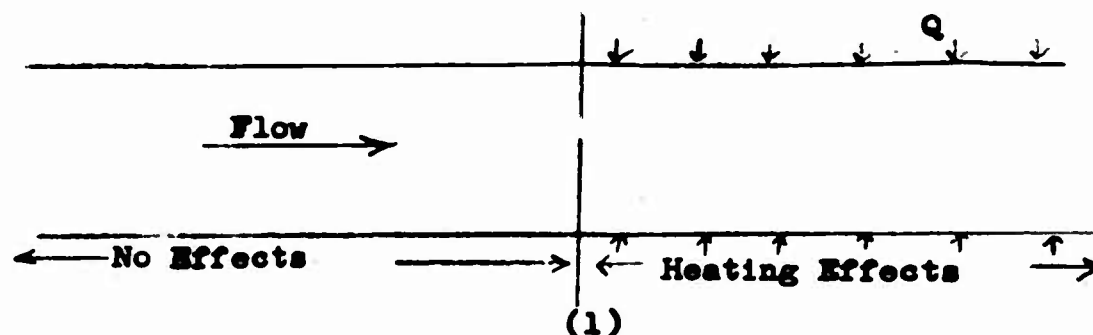
By simple  $T_0$  effects then we mean the following to obtain:

$$\begin{aligned} \text{constant area} & (dA = 0) \\ \text{no friction} & (dF = 0) \\ \text{no shaft work} & (dT_0 = \frac{dQ}{c_p}) \end{aligned}$$

Let us determine the locus of fluid states corresponding to simple  $T_0$  flow on the T-s diagram and discuss the characteristics of such a flow.

Rayleigh Line

Consider then the flow of a perfect gas in an adiabatic frictionless constant area duct. No variation of stream properties would exist in this flow. If, however, downstream of some station (1) the total temperature of the stream is caused to change by the presence of heating effects as depicted in the figure below then the stream properties will change. A relation between the stream properties at (1) and the pressure and temperature downstream of (1)



may be obtained by combining the momentum equation with the continuity equation\*. Thus,

$$\begin{aligned} p_1 + \rho_1 v_1^2 &= p + \rho v^2 \\ v &= \rho A V \end{aligned}$$

\*Compare following development with that of note 14, page 14.1. Notice the analogy with momentum equation here replacing the energy equation of note 14.

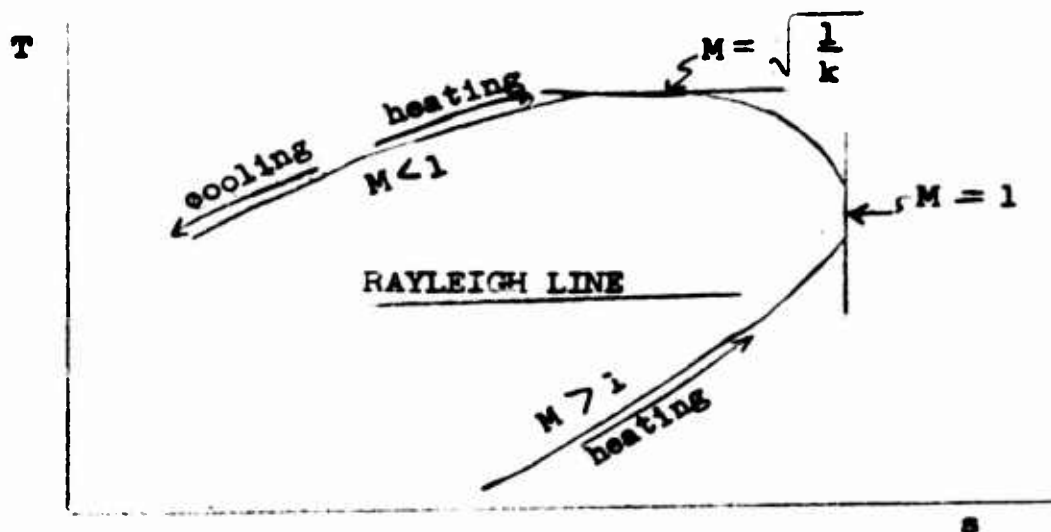
replacing  $V$  in momentum equation by  $\left(\frac{W}{A}\right)\left(\frac{1}{\rho}\right)$  from continuity equation gives, after using  $\rho = \frac{p}{RT}$ .

$$p_1 + \rho_1 v_1^2 = p + R\left(\frac{W}{A}\right)^2 \frac{T}{p}.$$

For given inlet conditions this equation represents a relation between pressure and temperature that must be satisfied at downstream points in the flow. By assuming values of pressure to exist downstream the corresponding values of temperature can be determined from the above relation. These values of  $p$  and  $T$  can then be used in the following equation to determine the required values of entropy

$$s = c_p \ln \frac{T}{T_1} - R \ln \frac{p}{p_1} + s_1.$$

The temperature entropy locus of such points that satisfy the continuity and momentum equations for simple  $T_0$  flow is called the Rayleigh line and is sketched below.



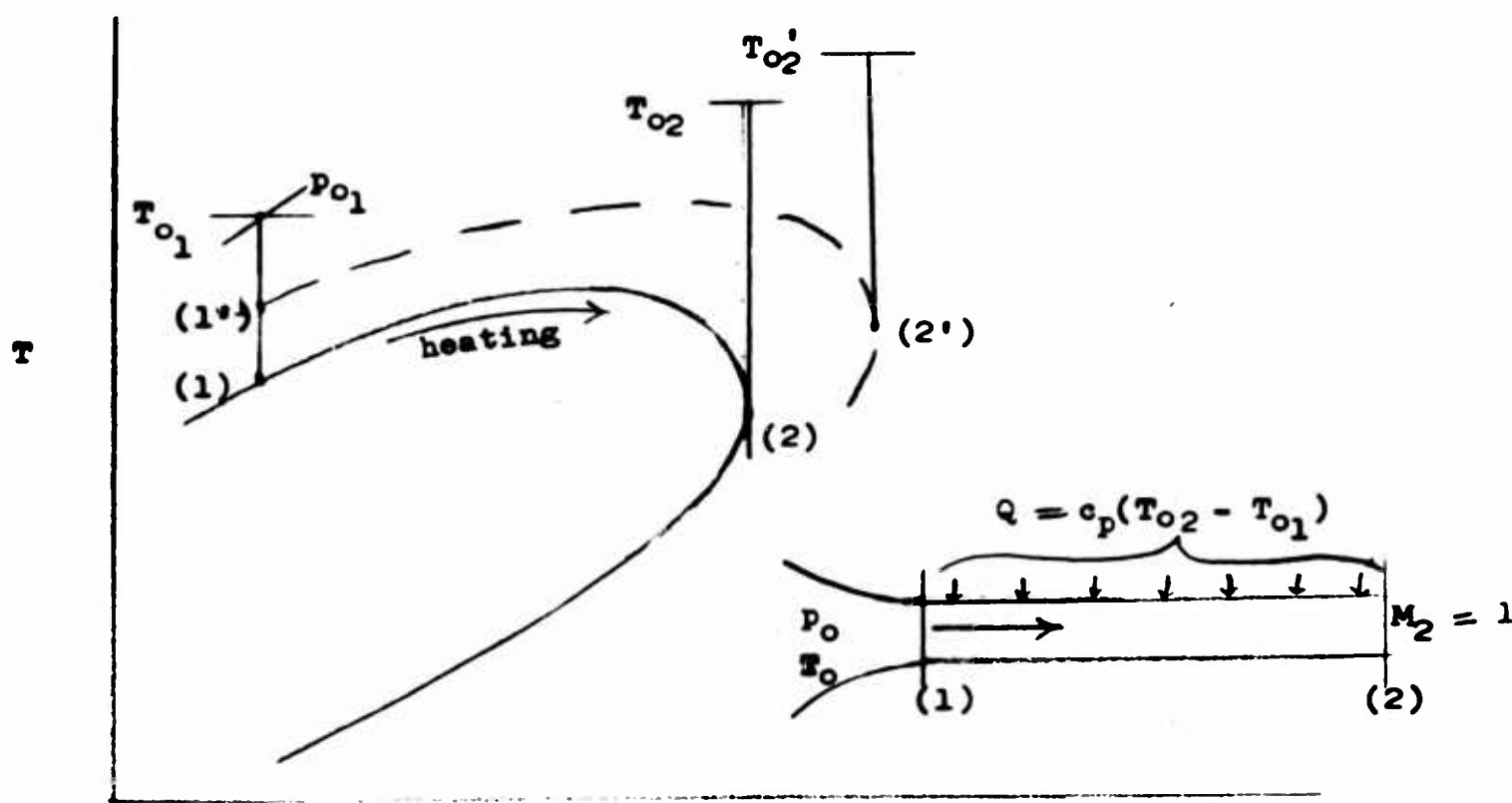
We can imagine the heating process in our flow to occur in a reversible manner so that between station (1) and any downstream station the entropy change is given by

$$s - s_1 = \int_{T_1}^T \left( \frac{dq}{T} \right)_{\text{Rev.}}$$

This indicates that heating produces a flow with downstream points at higher entropy and a heating process would therefore be in the direction shown on the temperature entropy diagram. Similarly a cooling process produces states of lower entropy.

The Rayleigh line indicates that it is impossible to go from a subsonic flow to a supersonic flow by a continuous heating process since heating beyond that point that gives  $M=1$  would require a decrease in entropy. It does appear however that heating the flow until a Mach Number of one is obtained followed by cooling would produce a transition from subsonic flow to supersonic flow. This situation is analogous to simple area flow in that a decrease of area to Mach of one followed by the reverse effect of an area increase produces a transition from subsonic flow to supersonic flow in simple area flow. It seems quite improbable, however, that one could obtain experimentally the transition from subsonic to supersonic flow with heating followed by cooling. The main reason being due to the fact that frictional effects in a real flow with heating can not be neglected.

Consider a subsonic simple  $T_0$  flow in which sufficient heating effects are present to produce sonic exit conditions from the duct. This process is shown on the figure below where (1) represents the inlet conditions to the duct and (2) indicates the exit condition corresponding to a Mach of one\*. What happens



\*It is assumed here that the exhaust region pressure is at the value required to give this result.

If there is heating in excess of that required to give sonic exit conditions? Experiments show that this further heating produces a readjustment in the flow which results in a reduced mass rate of flow with Mach one still maintained at the exit section of the duct. Thus the new flow process would lie on a Rayleigh line such as the dashed one in the figure with the flow proceeding from 1' to 2' with  $T_{02}' > T_{02}$ . Thus we find a choking phenomenon to occur in simple  $T_0$  flow as well as in simple area and frictional flow.

## Lecture Note 17

SIMPLE  $T_o$  FLOW - II

We consider in this note the problem of finding expressions relating the stream properties in a simple  $T_o$  flow to the flow Mach Number. The procedure to be followed is that which was used in notes 10 and 15. We will find relations such as

$$\left(\frac{T_o}{T_o^*}\right) = f_1(M)$$

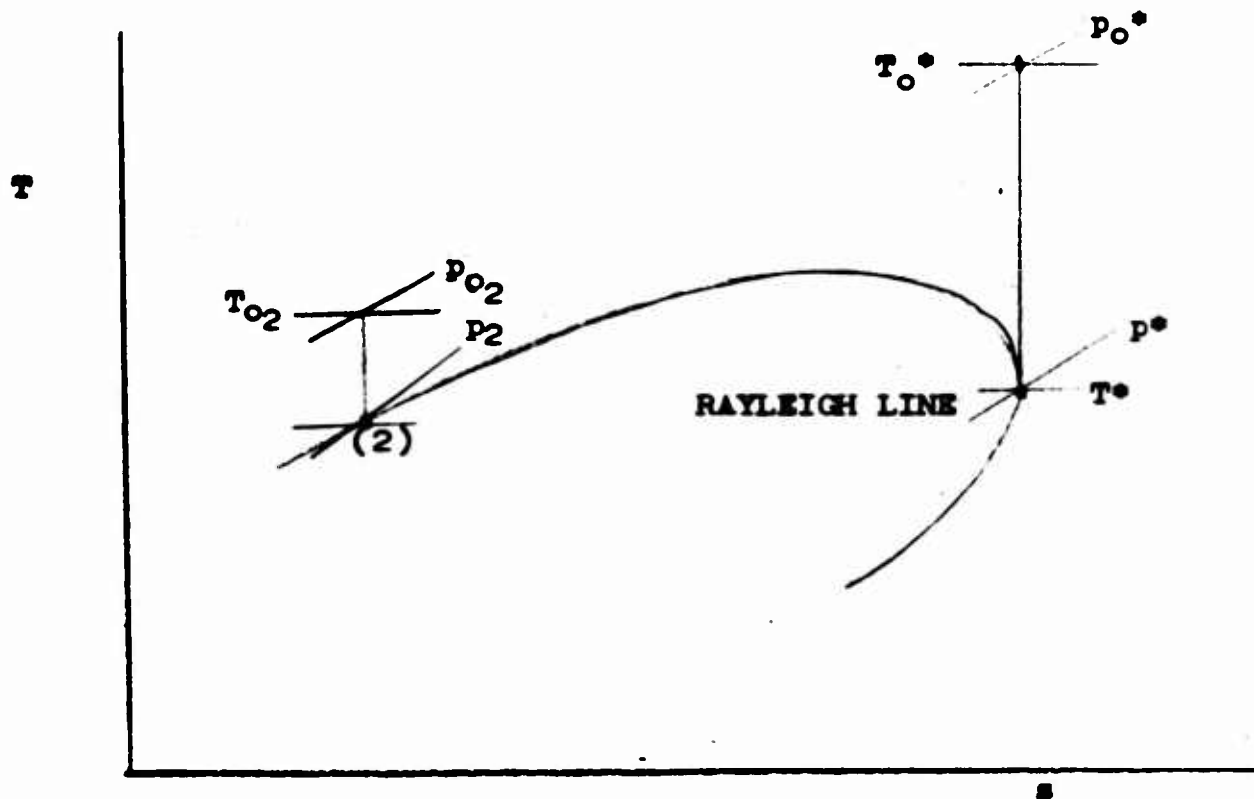
$$\left(\frac{p_o}{p_o^*}\right) = f_3(M)$$

$$\left(\frac{p}{p^*}\right) = f_2(M)$$

$$\left(\frac{T}{T^*}\right) = f_4(M)$$

where the starred quantities refer to the stream properties in simple  $T_o$  flow where the Mach Number is one. Thus referring to the temperature entropy diagram below we will find that  $T_o$  at (2) divided by  $T_o^*$  of the Rayleigh line is a function of  $M_2$  and similarly for the ratios

$$(p/p^*)_2, (p_o/p_o^*)_2, \text{ and } (T/T^*)_2$$



The following equations can be written for simple  $T_o$  flow:

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} \quad (\text{state})$$

$$\frac{dT}{T} + \frac{\frac{k-1}{2} M^2}{1 + \frac{k-1}{2} M^2} \frac{dM^2}{M^2} = \frac{dT_o}{T_o} \quad (\text{total temp definition})$$

$$\frac{d\rho}{\rho} + \frac{dV}{V} = 0 \quad (\text{continuity})$$

$$\frac{dp}{p} + \frac{kM^2}{1 + \frac{k-1}{2} M^2} \frac{dM^2}{M^2} = \frac{dp_o}{p_o} \quad (\text{total pressure definition})$$

$$\frac{dM^2}{M^2} = 2 \frac{dV}{V} - \frac{dT}{T} \quad (\text{Mach Number definition})$$

$$\frac{dp}{p} + \frac{kM^2}{1 + kM^2} \frac{dM^2}{M^2} = 0 \quad (\text{impulse function definition})$$

$$\frac{ds}{c_p} = \frac{dT}{T} - \frac{k-1}{k} \frac{dp}{p} \quad (\text{entropy definition})$$

These seven equations have the eight variables  $p$ ,  $\rho$ ,  $T$ ,  $M^2$ ,  $T_o$ ,  $V$ ,  $s$ , and  $p_o$ . Select  $T_o$  as independent and find by simultaneous solution of the above system of equations how the remaining stream properties depend upon the total temperature variation. The results are summarized in the following tabulation.

	$\frac{dT_o}{T_o}$
$\frac{dM^2}{M^2}$	$\frac{(1 + kM^2) (1 + \frac{k-1}{2} M^2)}{1 - M^2}$
$\frac{dp}{p}$	$-\frac{kM^2 (1 + \frac{k-1}{2} M^2)}{1 - M^2}$
$\frac{dp_o}{p_o}$	$-\frac{kM^2}{2}$
$\frac{dT}{T}$	$\frac{(1 - kM^2) (1 + \frac{k-1}{2} M^2)}{1 - M^2}$
$\frac{d\rho}{\rho}$	$-\frac{1 + \frac{k-1}{2} M^2}{1 - M^2}$
$\frac{dV}{V}$	$\frac{1 + \frac{k-1}{2} M^2}{1 - M^2}$
$\frac{ds}{c_p}$	$1 + \frac{k-1}{2} M^2$
Table is read: $\frac{dp}{p} = - \frac{kM^2 (1 + \frac{k-1}{2} M^2)}{1 - M^2} \frac{dT_o}{T_o}$	

Some of the more important conclusions that can be drawn from this table are:

$$(a) \quad \frac{dp_o}{dT_o} = - \frac{kM^2}{2} \frac{p_o}{T_o}$$

thus total pressure decreases with increasing  $T_o$  (heating) and  $p_o$  increases with decreasing  $T_o$  (cooling)

$$(b) \quad \frac{ds}{dT} = \frac{1 - M^2}{1 - kM^2} \frac{c_p}{T}$$

thus at  $M = 1$ ,  $\frac{ds}{dT} = 0$  and entropy is maximum at  $M = 1$

$$(c) \quad \frac{dM}{dT_o} = \frac{f(M, T_o)}{1 - M^2}$$

thus effect of heating on Mach Number is of opposite sign in subsonic and supersonic flow.

Other conclusions can similarly be drawn with the aid of the differential relations tabulated.

$T_o/T_o^*$ ,  $p_o/p_o^*$  etc. as functions of Mach Number

Notice that we have found above the relation

$$\frac{dM^2}{M^2} = f(M) \frac{dT_o}{T_o}$$

or

$$\frac{dT_o}{T_o} = f_1(M) \frac{dM^2}{M^2}$$

This result may be integrated to give  $T_o/T_o^*$  in simple  $T_o$  flow as a function of Mach Number. Similarly we have from the above table

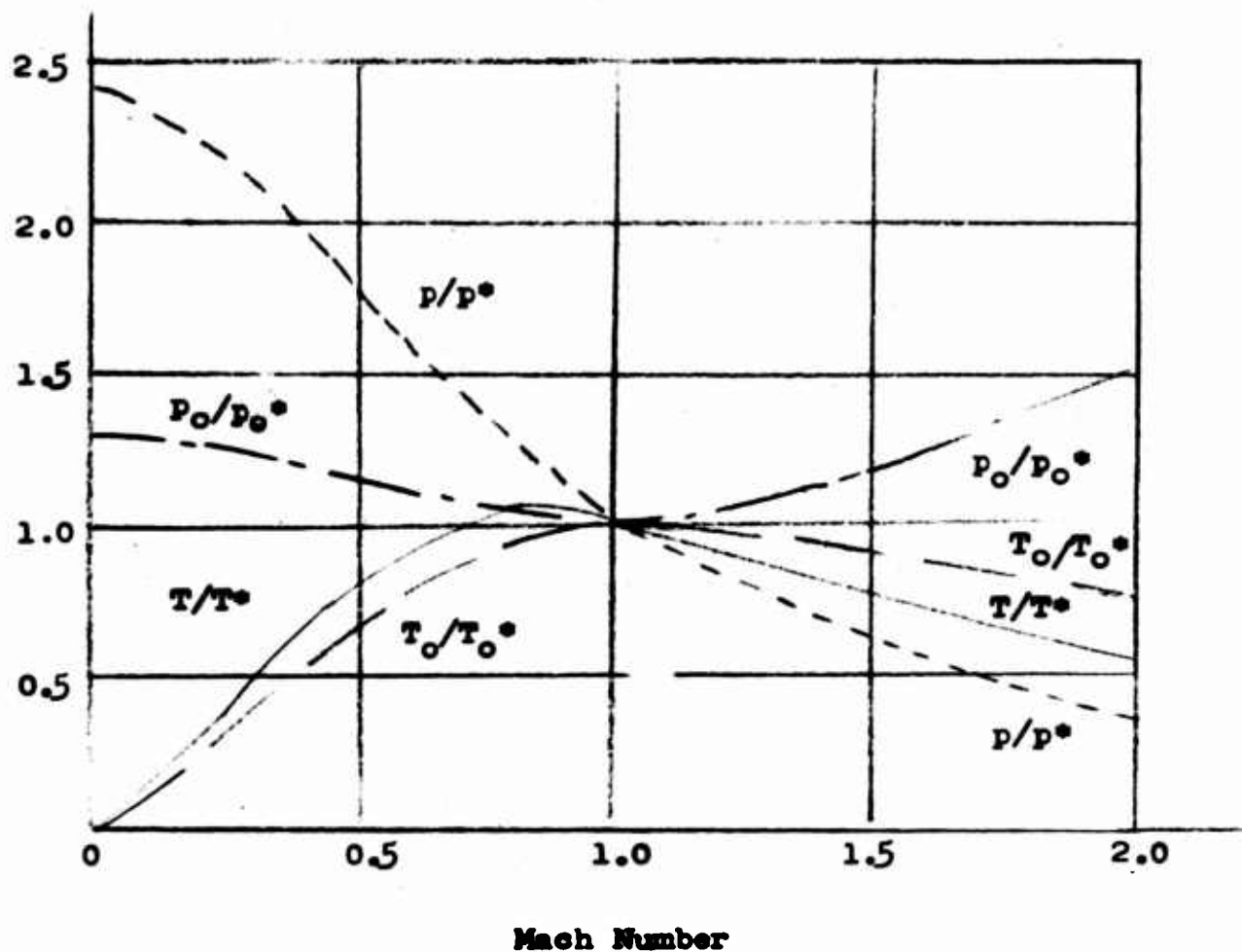
$$\frac{dp_o}{p_o} = f_2(M) \frac{dT_o}{T_o}$$

or

$$\frac{dp_o}{p_o} = f_2(M) f_1(M) \frac{dM^2}{M^2}$$



which when integrated from  $p_0$  to  $p_0^*$  and  $M$  to  $M = 1$  gives  $p_0/p_0^*$  in terms of Mach Number. Proceeding along these lines one can get the relations given on page 210 of the Gas Tables by Keenan and Kaye. These relations are tabulated in the Gas Tables on page 148 and following. A plot of the tabulated stream properties versus Mach Number is given below.



Notice that the static temperature ratio reaches a maximum just to the left of  $M = 1$ . This corresponds to the maximum point attained on the subsonic branch of the Rayleigh line.

## Lecture Note 18

## SUPERSONIC DIFFUSERS

Diffusers, or passages which decelerate the stream to low velocity, are important elements in such different devices as compressors, wind tunnels, and ram jets. The supersonic diffuser offers certain unusual problems not met with in the design of subsonic nozzles.

At first thought it might appear that a supersonic diffuser could be designed as though it were the reverse of a converging-diverging nozzle. Two difficulties arise, however. First, if there is a supersonic nozzle within the system, it is virtually impossible to design properly the throat of the diffuser because frictional effects between the nozzle and the diffuser require that the diffuser throat be larger than the nozzle throat. If the diffuser throat is made slightly too small, supersonic flow will not be attained in the nozzle; and, if the diffuser throat is made slightly too large, there will necessarily be a shock somewhere within the diffuser. Indeed, even if the two throats did match perfectly, it appears that the combine system would be unstable.

A second and more serious difficulty arises. Most flow systems start from rest and accelerate to the operating velocity. If we focus attention on a supersonic wind tunnel as a particular example (Figure 1) the discussion of Lecture Note 9 indicates that a shock will move down through the nozzle as the pressure ratio across the nozzle is increased. However, a normal shock reduces the stagnation pressure of the stream. It is evident from the relation

$$\frac{w\sqrt{T_0}}{A^*P_0} = f(M) \text{ evaluated at } M = 1.0, \text{ i.e., } \frac{w\sqrt{T_0}}{A^*P_0} = 0.0166 \frac{\text{slug}^{1/2}\text{OR}}{\text{Sec} - \text{lbF}}$$

the product of minimum area and total pressure is constant for a constant flow rate and stagnation temperature. Thus

$$A^*P_0 = \frac{w\sqrt{T_0}}{0.0166} = \text{constant} \quad (1)$$

During the period, therefore, when the shock passes through the nozzle and test section, the diffuser throat must be larger than the nozzle throat. The minimum ratio of the two areas necessary for starting corresponds to the condition of greatest loss in stagnation pressure, that is, to the condition when the shock is in the test section. Ignoring frictional effects, the minimum ratio of diffuser throat area to nozzle throat area is found by using Equation 1. Referring to Figure 1, we get

$$\left(\frac{A^*_y}{A^*_x}\right)_{\min} = \left(\frac{P_{0x}}{P_{0y}}\right)_{(M_x)_{\max}} \quad (2)$$

The limiting contraction ratio for the diffuser, that is the minimum value of  $A_{\text{diff. throat}}/A_{\text{diff. inlet}}$  is shown in Fig. 2. For comparison, the contraction ratio for isentropic diffusion to Mach Number unity is also shown.

At the limiting condition, the diffuser is barely able to "swallow" the shock and the Mach Number at the diffuser throat is unity when the shock is in the test section. If the diffuser throat is smaller than required by (Eq. 2) either a normal shock will stand in the diverging portion of the nozzle or there will be no supersonic region at all in the nozzle.

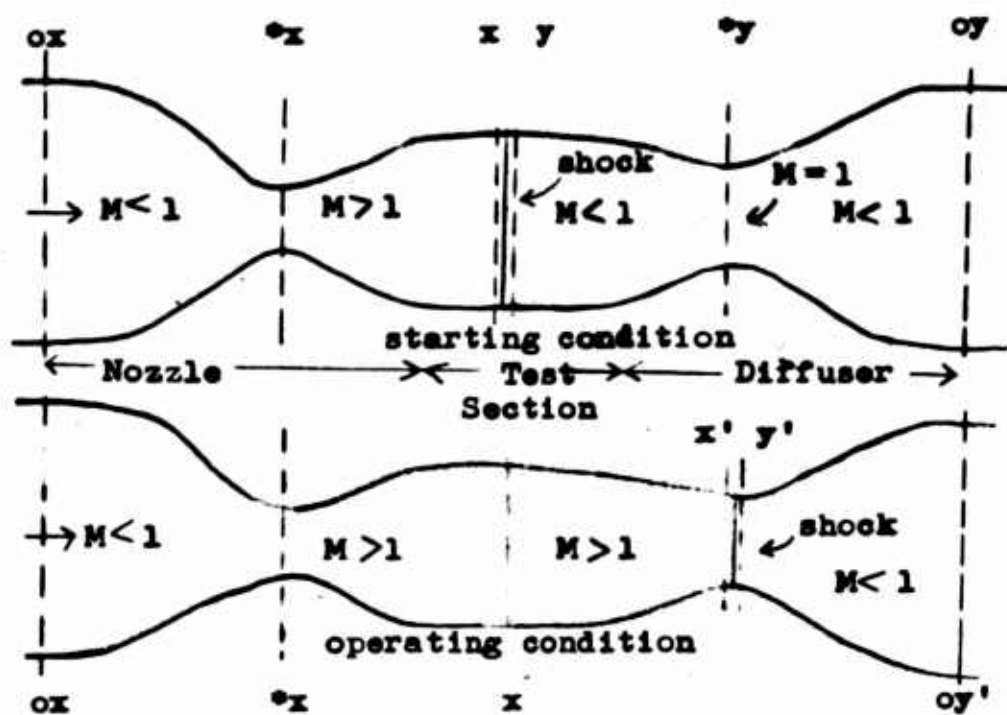


Figure 1

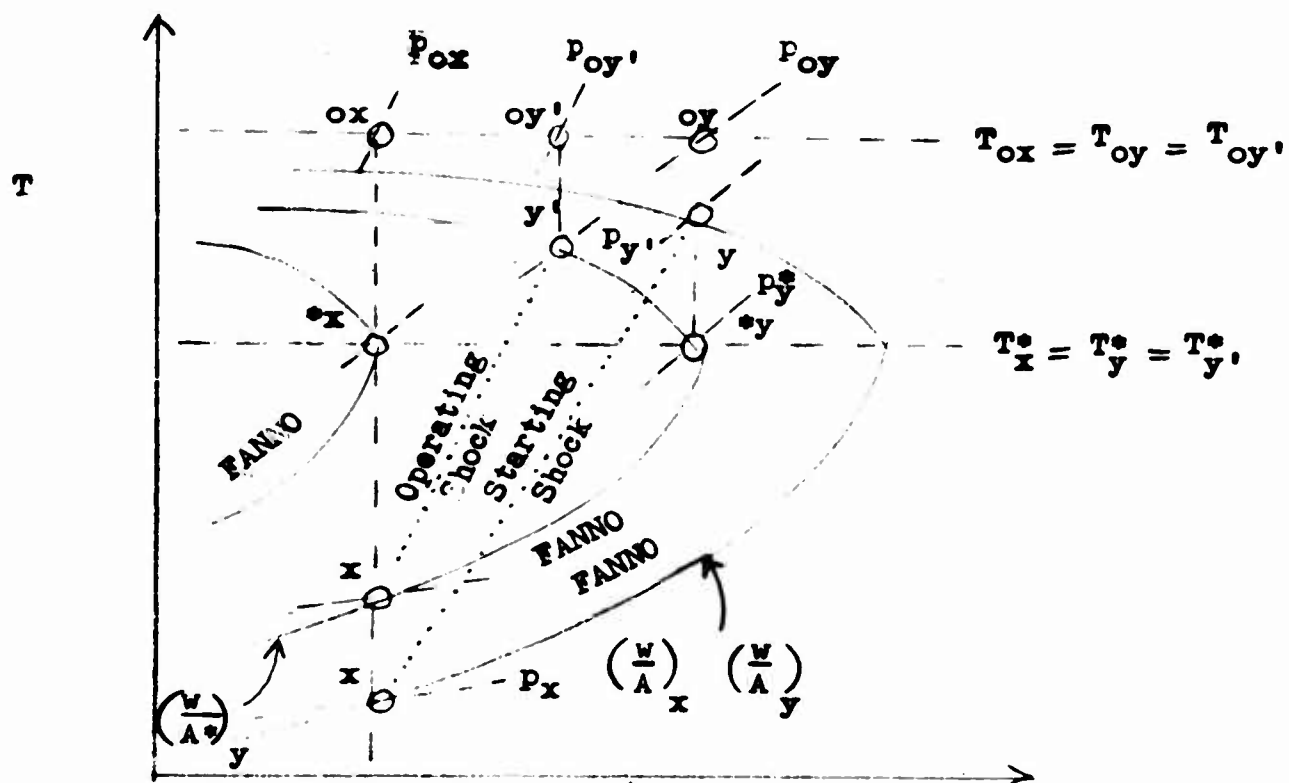


Figure 3

Contraction Ratio  $C_R$  =  $\frac{A_{diff. throat}}{A_{diff. inlet}}$

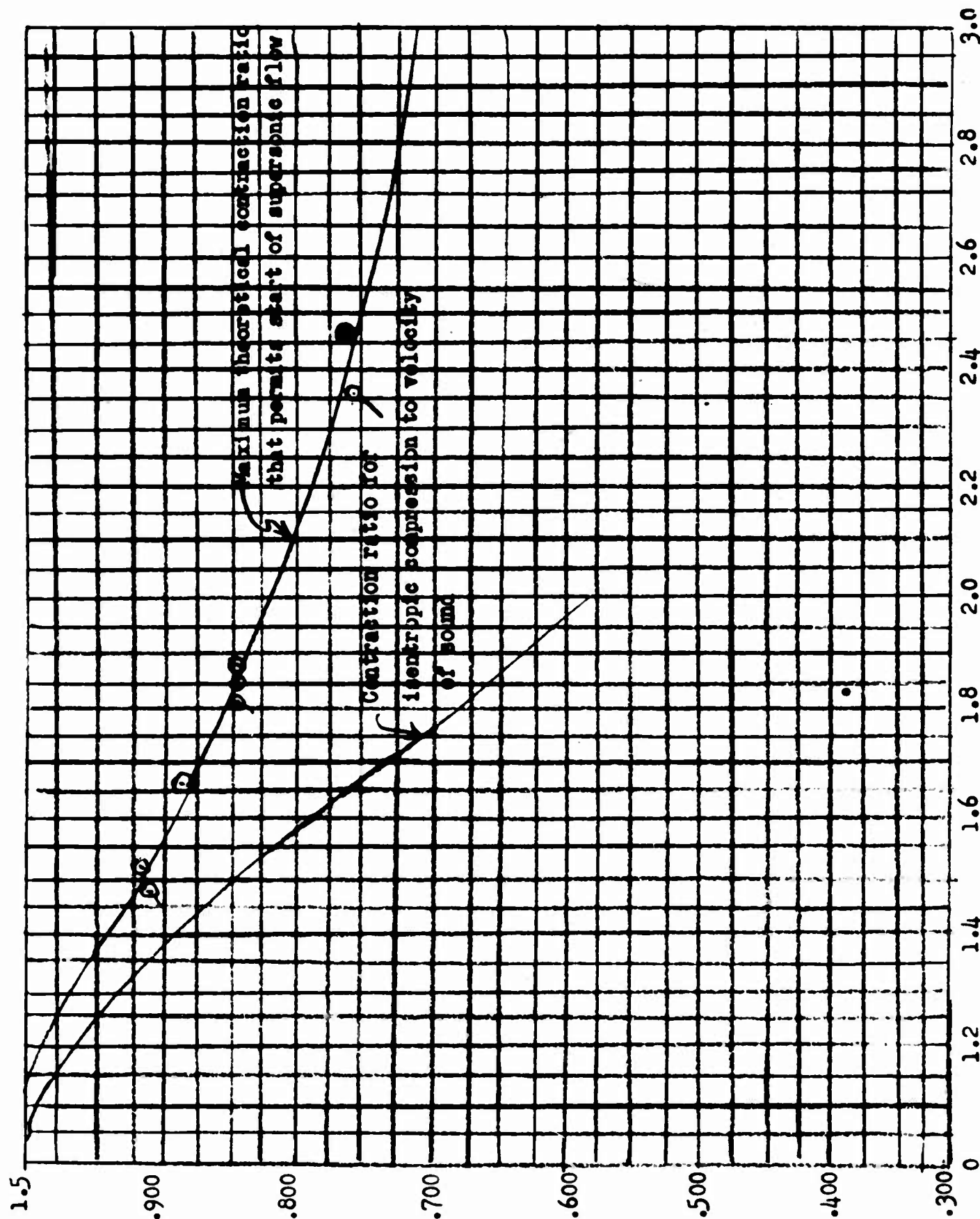


Figure 2 - Contraction ratio permissible for constant-geometry supersonic diffusers. Tailed symbols represent Mach numbers at which the diffusers failed to operate; Plain symbols represent minimum Mach numbers at which diffusers operated. (From NACA Wartime Rpt. L-713, "Preliminary Investigation of Supersonic Diffusers" by Arthur Kantrowitz and Coleman duP. Donaldson)

These ideas are illustrated graphically in the T-s diagram, Figure 3. In interpreting this diagram it is well to remember that all states on the same Fanno line have a common stagnation temperature and flow per unit area. A change in cross-sectional area has associated with it a shift from one Fanno line to another. It is clear from this diagram that during the "starting" condition there is a large loss in stagnation pressure and a consequent increase in the area required to pass the flow. The path of states during the limiting starting condition (at least while the shock is in the test section) is from x to y to  $\infty$  to oy.

Assuming that the diffuser throat is made sufficiently large, the shock will be able to enter the diffuser. Its position during operating conditions will depend on the back pressure on the diffuser. From the standpoint of efficiency, the shock should be maintained at the diffuser throat, for the shock will then occur at the minimum Mach Number in the diffuser. The best design for a one-dimensional, supersonic diffuser of fixed geometry has, therefore, a minimum area barely large enough to pass the flow during starting conditions, and has the shock at the minimum area during operating conditions. The best starting and operating conditions are shown in Figure 1 and 3. During operation, the path of states in Figure 3 is from x to x' to y' to oy'.

In practice, the shock is maintained slightly downstream of the throat during operation. This is done because, with a fixed back pressure, the shock is unstable in the converging portion of the diffuser. For example, if the shock were maintained exactly at the minimum area, a slight disturbance might make it move temporarily into the converging section. But this would augment the loss in stagnation pressure, and, if the back pressure were fixed, the shock would move further upstream. This would make the situation still worse, and the shock would move upstream progressively until it came to rest in the nozzle at a point where the stagnation-pressure loss in the system matched the back pressure on the system. In order again to obtain supersonic flow in the test section, it would be necessary to lower the back pressure to the minimum value required for starting.

To insure that a supersonic diffuser of fixed geometry will start, the throat must be made slightly larger than the limiting value to account for inaccurate estimates of the effects of friction, of the departures from one-dimensionality, and so forth.

Thus, because practical considerations require that the best design be comprised by an enlargement of the throat and by an operating condition with the shock at a Mach Number greater than the minimum in the passage, the practically attainable efficiencies of such diffusers fall short of the values which seem possible in principle.

The loss in stagnation pressure during operation is much less than during starting, as shown by Figure 3. In the case of a wind tunnel, this reduces the power consumption during operation, but the pressure ratio of the compressors and the maximum power are determined by the starting conditions. Thus, as compared with a simple shock-type diffuser, the contraction-type diffuser is of advantage only in that it reduces the power expenditure during operation.

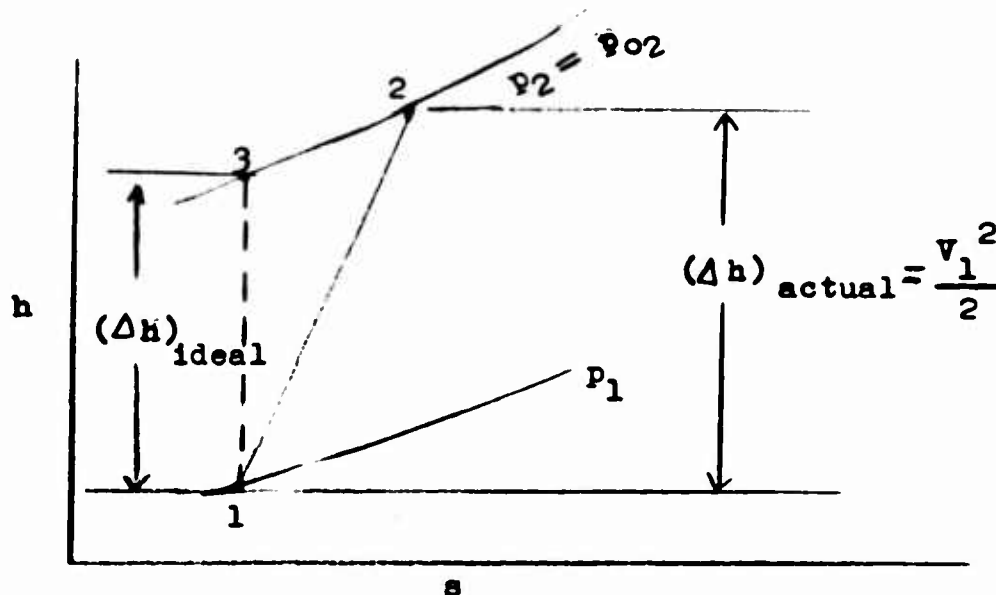


Figure 4

Some of the starting difficulties mentioned above may be avoided through the use of diffusers with adjustable throats, by temporarily overspeeding the stream upstream of the throat, by pushing the shock through the throat with a large pressure pulse, or using oblique-shock diffusers.

The most common definition of diffuser efficiency is parallel to the definition employed for compressor efficiency. Referring to Figure 4 and assuming that the velocity leaving the diffuser is negligible, we define

$$\eta_D = \frac{(\Delta h)_{\text{ideal}}}{(\Delta h)_{\text{actual}}} = \frac{h_3 - h_1}{h_2 - h_1} \quad (3)$$

where state 1 is the actual state entering the diffuser, 2 is the actual state leaving the diffuser, and 3 is a fictitious state at the actual leaving pressure but at the entering entropy. For a perfect gas Equation 3 becomes:

$$\eta_D = \frac{T_3 - T_1}{T_2 - T_1} = \frac{T_1 (T_3 - 1)}{T_2 - T_1} = \frac{T_1 (T_3 - 1)}{v_1^2 / 2c_p}$$

and, since  $T_3/T_1 = (p_2/p_1)^{\frac{k-1}{k}}$

and  $T_1 = a_1^2 / kR$

we get, 
$$\eta_D = \frac{(p_2/p_1)^{\frac{k-1}{k}} - 1}{\frac{k-1}{2} M_1^2} \quad (4)$$

Variable area supersonic diffuser

Consider now, as a further illustration of the supersonic diffuser operating characteristics, the flow through the arrangement of Figure 1 as the diffuser throat area is increased from zero area up to and beyond the area required to start the diffuser. It will be convenient and helpful to show the variation of diffuser throat area on a graph of area ratio versus inlet Mach number as given in Figure 5\*.

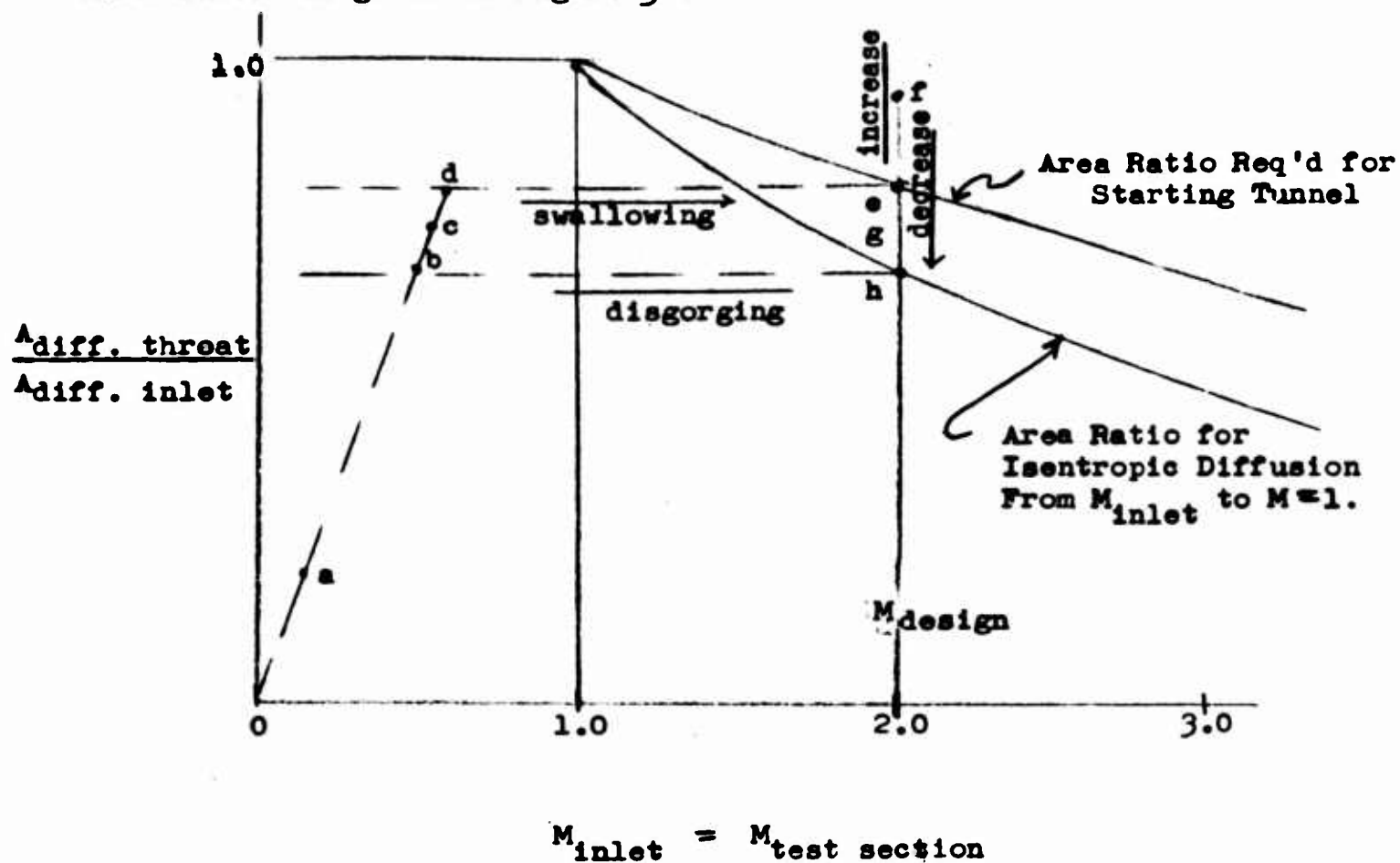
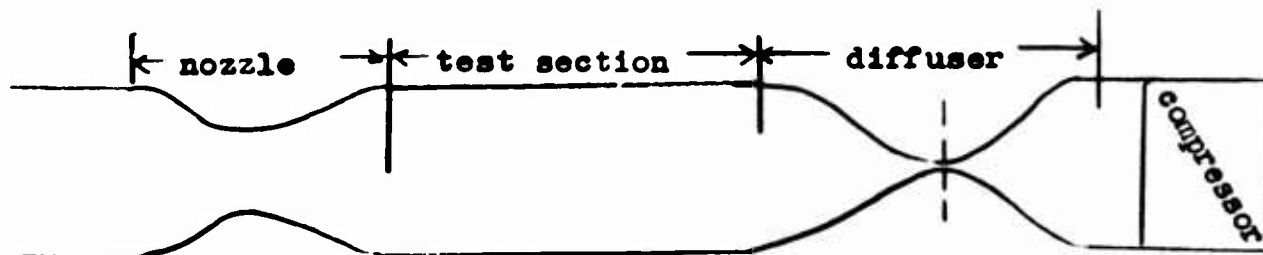


Figure 5

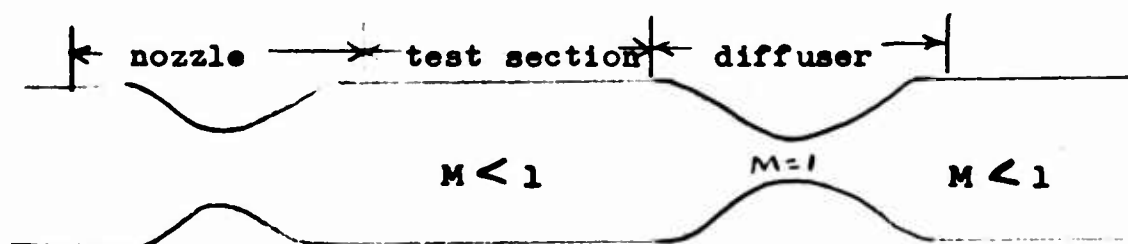
\*Figure 5 is not to scale also Figure 4 is Figure 2 with the abscissa extended to zero.

In Figure 5 the variation of the diffuser inlet Mach Number as the diffuser throat area increases is along the line from (0) to (d) thence to (e) and (f). The variation of inlet Mach number as the throat area decreases from (f) is along the line (f) to (h) thence to (b) and back to zero. In the discussion to follow we will assume that the back pressure is adjusted to the value required to give the condition described through the unit.



case (0) - no flow

For the case of zero diffuser throat area no flow exists and  $M=0$  throughout the unit. Now as the throat area is increased from a zero value the flow throughout the unit is subsonic preceeding and following the diffuser throat with  $M=1$  at this throat. This condition (case a) exists until the diffuser throat area becomes equal to the nozzle throat area and is depicted schematically below.



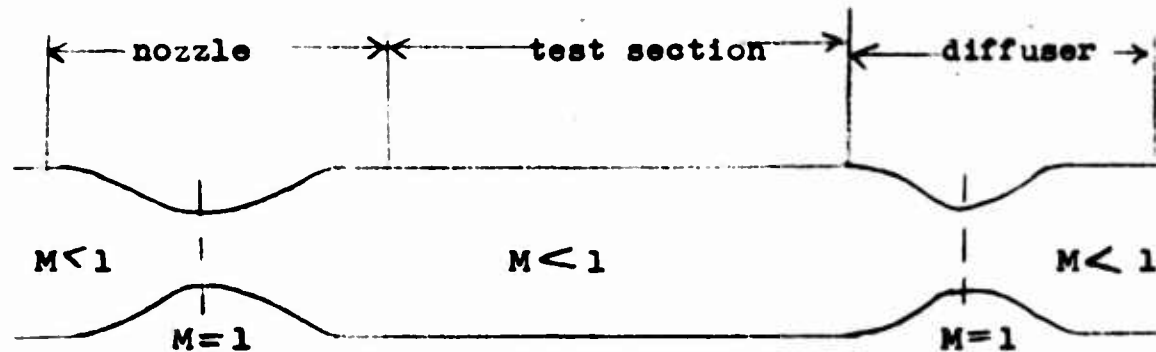
case (a) -  $A_{\text{diff. throat}} < A_{\text{nozzle throat}}$



When the diffuser throat area equals the nozzle throat area sonic flow will exist at each throat with  $M < 1$  elsewhere. Assuming the test section is built for a design Mach number of 2 the throat areas will be equal when

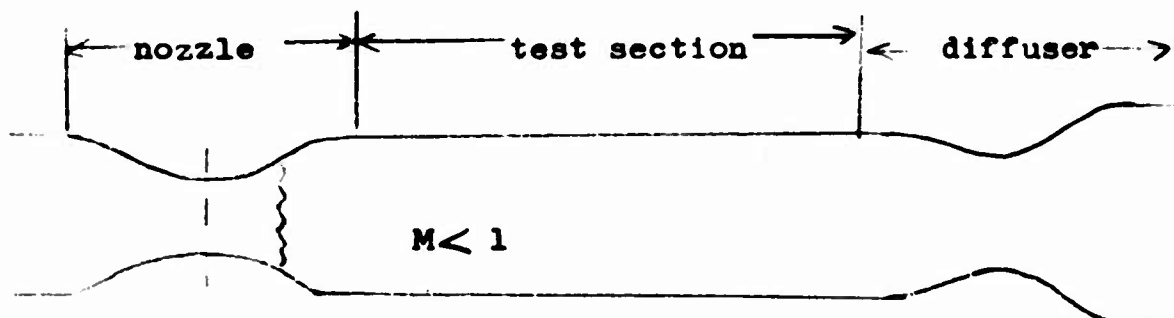
$$\frac{A_{\text{diff. throat}}}{A_{\text{Diff. inlet}}} = \left(\frac{A^*}{A}\right)_{M=2} = 0.595.$$

Case b is shown herewith.



$$\text{case (b)} - A_{\text{diff. throat}} = A_{\text{nozzle throat}}$$

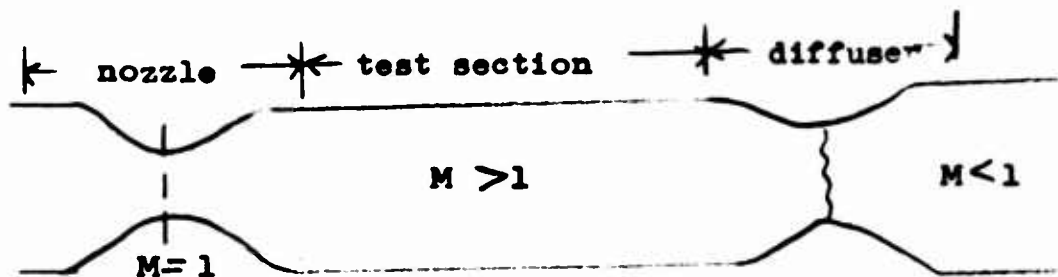
With the area increased beyond that of case (b) a normal shock occurs in the divergent section of the nozzle and moves downstream to the test section as the area increases from condition (b) to (d). An intermediate condition between (b) and (d) is illustrated in the following figure.



$$\text{case (c)} - \left(A_{\text{diff. throat}}\right)_{\text{starting}} > \left(A_{\text{diff. throat}}\right)_c > A_{\text{nozzle throat}}$$

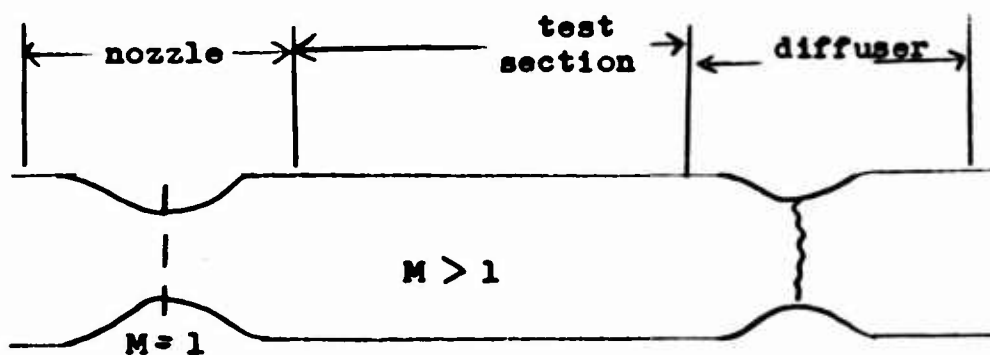
An infinitesimal increase in the diffuser throat area beyond that area (d) for which the normal shock is in the test section gives a swallowed shock as the inlet Mach number goes from a sub to a supersonic value and the diffuser is started with a normal shock at its throat. A further increase in throat area from (e) to (f) does not affect the inlet Mach number but does, of course, increase the value of the diffuser throat Mach number.

Condition (f) is shown below



condition (f) -  $A_{\text{diff. throat}} > (A_{\text{diff. throat}})_{\text{starting}}$

If, after the diffuser has started, we decrease the diffuser throat area from (f) to a value (g) we find that the inlet Mach number remains constant at the design value. As far as geometry is concerned case (g) and case (e) are identical. Thus for the case (g) we have schematically, the following.



case (g) -  $(A_{\text{diff. throat}})_{\text{starting}} > (A_{\text{diff. throat}})_g > A_{\text{nozzle throat}}$

Although case (e) and case (g) are identical geometrically, the flow processes existing for the two cases differ markedly. This results from the fact that the geometry corresponding to (e) and (g) gives a different flow depending upon whether that geometry is approached during the establishment of the supersonic inlet (when shock waves of necessity occur ahead of the diffuser throat) or whether that geometry is attained after the establishment of the supersonic inlet flow. In the latter case no shocks precede the diffuser throat. Thus is it possible to pass the supersonic flow through the diffuser throat as long as its area is equal to or greater than the nozzle throat area since the total pressure remains constant between these throats with no intermediate shocks.

When the diffuser throat area is decreased to  $h$ , sonic flow occurs in each throat. With an infinitesimal decrease in diffuser throat area from condition (h) a shock arises immediately ahead of the diffuser throat and advances into the oncoming flow until condition (b) is attained as depicted schematically above. Further decrease in diffuser area reduces the diffuser inlet Mach number from (b) to (a) to zero.

#### Fixed geometry supersonic engine inlet.

The operating characteristics of a supersonic diffuser when used as an engine inlet can be illustrated in a manner similar to the discussion of the preceding section. Consider the operating conditions of a supersonic inlet of fixed geometry as the engine flight Mach number is increased from zero up to its design value. In Figure 5 we show the variation of diffuser inlet.

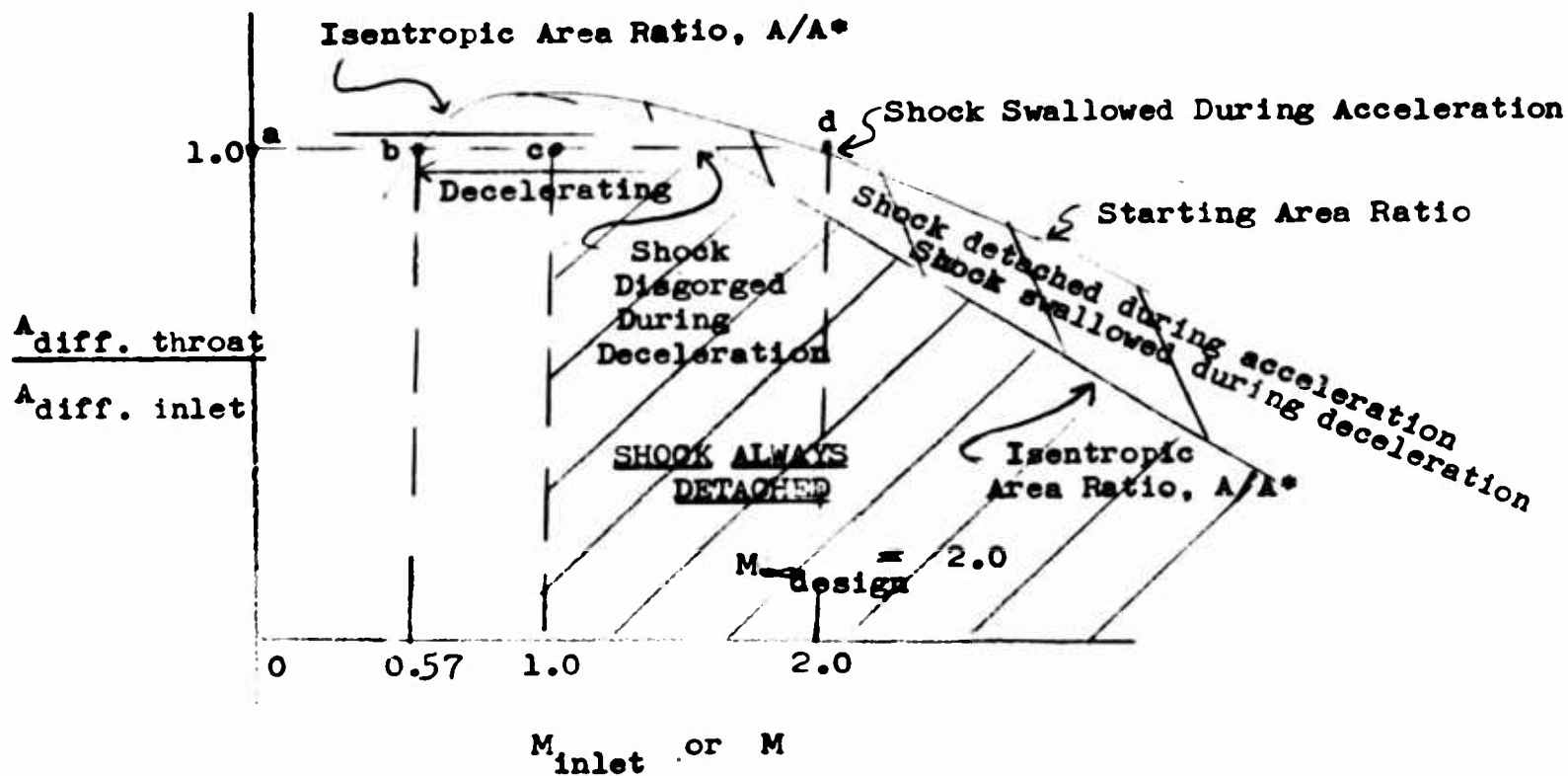
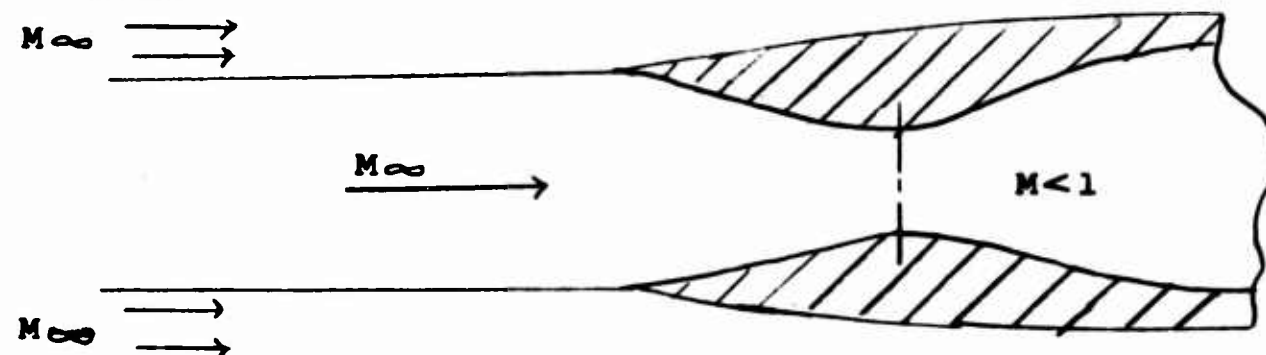


Figure 4

Mach number with the area ratio ( $A_{\text{diff. th.}}/A_{\text{diff. inlet}}$ ). Notice in the discussion to follow that the diffuser inlet Mach number during some operating conditions is identical with the free stream Mach number,  $M_\infty$ . In general, however, the flight or free stream Mach number is not equal to the engine inlet Mach number.

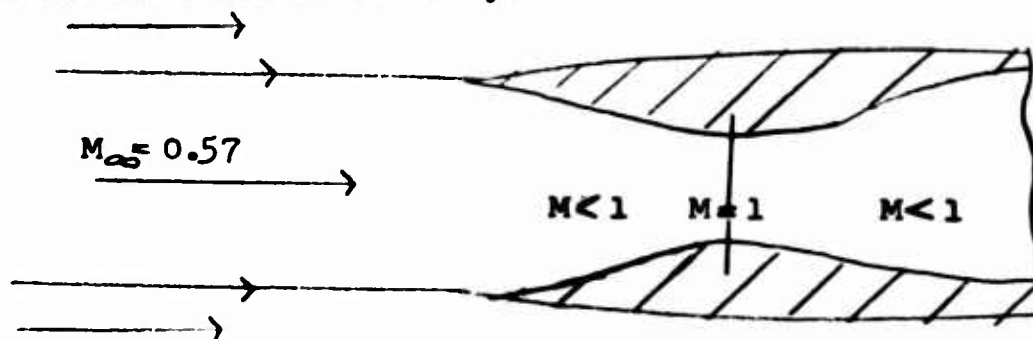
We will examine the phenomena in the diffuser as the Mach number,  $M_\infty$ , of the free stream is brought up to a value equal to and then greater than the diffuser design value of 2.0. Later the phenomena will be discussed as  $M_\infty$  is decreased from a value greater than 2.0 to a zero value. It will be assumed throughout that the back pressure on the diffuser is such to give the operating condition specified at any instant.

For a design value of  $M_\infty = 2$  the diffuser area ratio required for starting is 0.822 as given in Figure 2. At zero free stream Mach number there is no flow through the unit and  $M_{\text{inlet}} = 0$  corresponding to (a) to (b). During this interval of operation we have subsonic flow throughout the diffuser with  $M = M_{\text{inlet}}$  as depicted below



case (a) to (b) -  $M_{\text{inlet}} = M_\infty$  (back press is held at proper value to give this state of affairs)

When the free stream Mach number becomes equal to the subsonic  $M$  corresponding to an  $A/A^*$  equal to ( $A_{\text{inlet}}/A_{\text{throat}}$ ), condition (b) exists and the throat Mach number is unity.



Case (b) -  $M_\infty = M_{\text{inlet}}$ ; Mach number corresponding to an  $A/A^*$

$$\frac{A_{\text{inlet}}}{A_{\text{throat}}} = 1.22 \text{ is } M_{\text{inlet}} = 0.57$$

This case is shown above.

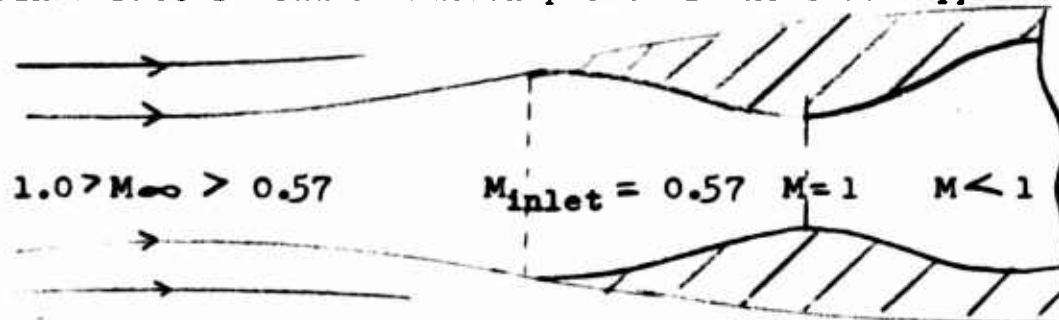
Since  $\frac{A_{throat}}{A_{inlet}} = 0.822$  we have for case (b) an  $\frac{A_{inlet}}{A^*} = \frac{1}{0.822} = 1.22$  so that

$M_\infty = M_{inlet} = 0.57$  as found from the isentropic tables.

As the free stream Mach number is now increased beyond  $M_\infty = 0.57$  the inlet Mach number remains at 0.57 with  $M = 1$  in the diffuser throat. The inlet Mach number must remain at 0.57 since  $A/A^* = \frac{A_{inlet}}{A_{throat}} =$  a fixed value. Thus

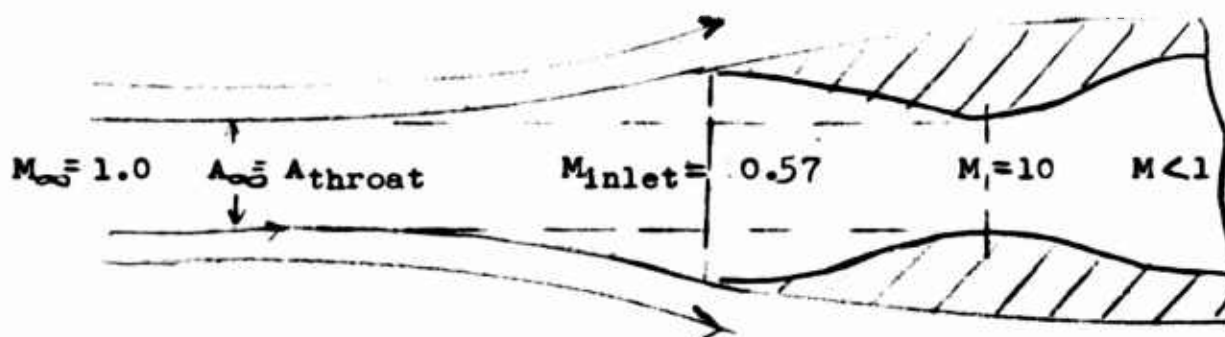
$A_{throat}$

between a free stream Mach number of 0.57 and 1.00 the condition shown herewith, wherein a free stream diffusion precedes the inlet applies.



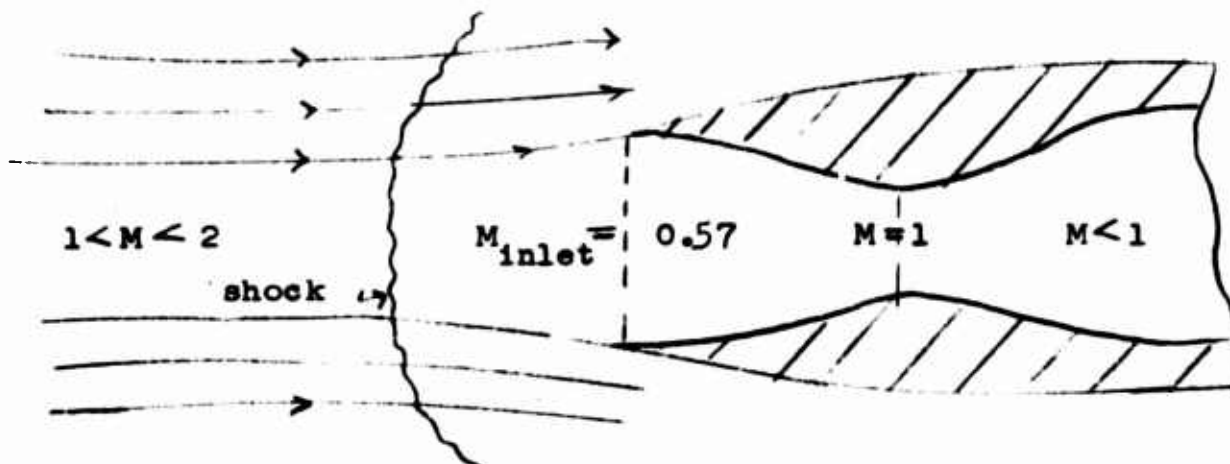
Case (b) to (c) -  $1 > M_\infty > 0.57$  (A free stream deceleration from  $M$  to  $M_{inlet}$  occurs ahead of engine as stream tube diverges.)

When the Mach number of the free stream becomes equal to one the area of the stream tube which handles the air going into the engine is equal to the engine diffuser throat area as indicated below for case (c).



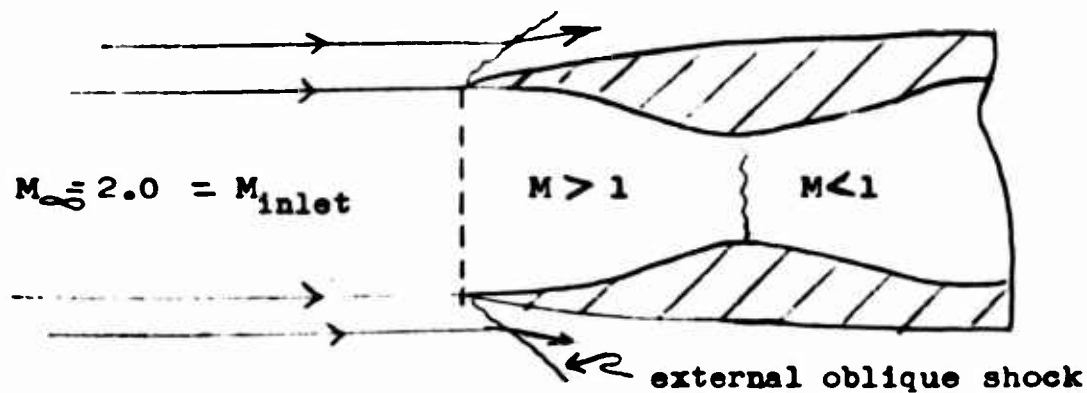
Case (c) -  $M_\infty = 1$  (Free stream tube area carrying air that enters engine diffuser throat area)

With a free stream Mach number greater than one a normal shock stands in front of the inlet until  $M$  is increased up to the design value of 2.0. The intermediate condition is shown below.



Case (c) to (d) -  $1.0 < M_{\infty} < 2$

When  $M = 2.0$  the diffuser starts and the shock is swallowed coming to rest in the throat of the diffuser - sketch below.



case (d) -  $M_{\infty} = 2.0$

As  $M_\infty$  becomes greater than 2.0 the shock remains swallowed if the back pressure is maintained sufficiently low.

Summarizing the operating characteristics of the inlet during the acceleration of  $M_\infty$  from 0 to 2.0 we have

- (a) to (b) -  $M_\infty = M_{\text{inlet}}$
- (b) to (c) -  $M_\infty$  increases and  $M_{\text{inlet}}$  remains constant at a value corresponding to  $M_b$ .
- (c) to (d) -  $M_\infty$  increases with normal shock occurring ahead of inlet and  $M_{\text{inlet}} = M_b$ .
- (d) and above - shock is swallowed.

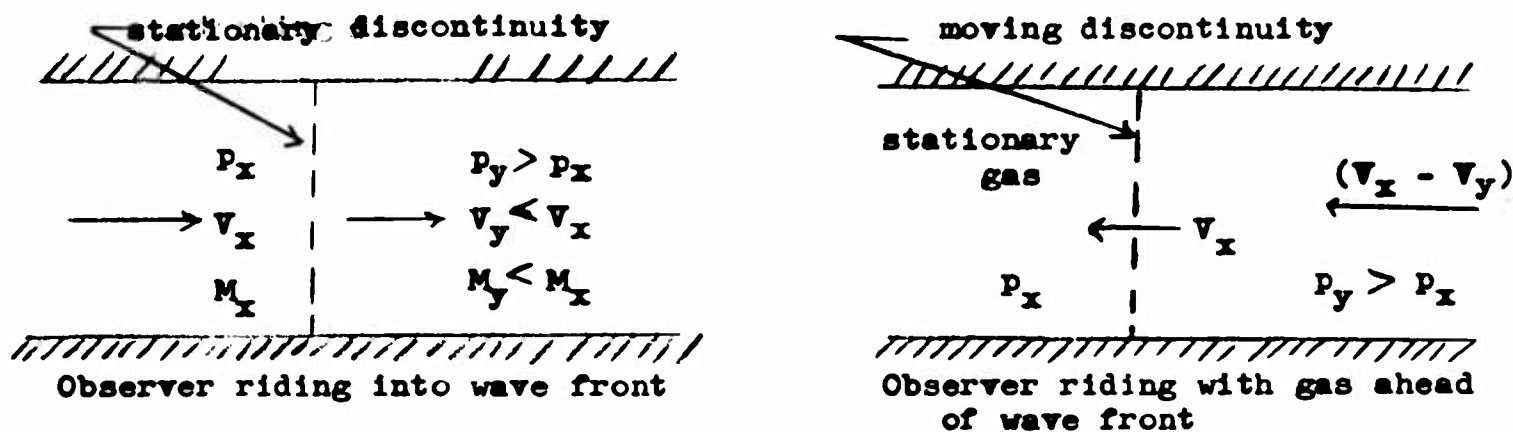
With a deceleration of  $M_\infty$  from  $M_\infty = 2.0$  to (e) the shock remains swallowed until  $M_\infty$  equals the supersonic value of  $M$  corresponding to  $A/A^* = 1.22$  at (e) or from isentropic tables when  $M = 1.56$ . With  $M_\infty = 1.56$  the shock is disgorged and  $M_{\text{inlet}}$  assumes a value of  $M_b = 0.57$ , with a normal shock occurring in front of the inlet until  $M_\infty$  becomes less than one. As the free stream Mach number becomes less than one we have a progression of states from (c) to (b) to (a) with the conditions already described for the acceleration from (a) to (c) applying.

## Lecture Note 19

Steadily Moving Shock Waves

Problems in which a pressure wave is moving at a uniform rate through a fluid initially at rest may be handled by using the methods of lecture note 12, on the normal shock wave. The equations of note 12 are applicable if the observer moves with the normal shock wave and if the quantities in the boxed equations refer to quantities relative to the moving observer. This involves changes in only those quantities which contain a velocity term, e.g., Mach number, stagnation temperature and stagnation pressure. Note that the static pressure, stream temperature, and sound velocity are the same for either observer. Also the discontinuity moves with a velocity  $V_x$  relative to the observer at rest.

The figures below show the steady flow through a discontinuity which is fixed relative to the observer and also the discontinuity advancing into air at rest relative to the observer. The steady flow through the stationary wave front may be transformed to the pattern of the moving discontinuity



by imagining the observer moves with the low pressure gas. This observer sees the wave front moving to the left with a speed  $V_x$ , and he sees the pressure in the stationary gas rise from  $p_x$  to  $p_y$  as the wave front advances into the stationary gas. The gas behind the wave front travels toward the front with a velocity  $(V_x - V_y)$ , and, since this is less than  $V_x$ , a particle of high-pressure gas falls further and further behind the front.

In order to make the boxed equations of note 12 applicable for an observer at rest with respect to the gas preceeding the wave front, all quantities containing a velocity term must be modified in accord with the change in coordinate system. Suppose we denote by primes those quantities measured relative to an observer who is at rest with respect to the gas preceeding the discontinuity. Then we may write

$$\begin{aligned} P_x' &= P_x; & P_y' &= P_y \\ T_x' &= T_x; & T_y' &= T_y \end{aligned}$$



$$M_x = V_x/a_x$$

$$M_x' = 0$$

$$M_y = V_y/a_y$$

$$M_y' = \frac{V_y - V_x}{a_y} = M_y - \frac{a_x}{a_y} M_x$$

$$T_{0x} = T_x \left(1 + \frac{k-1}{2} M_x^2\right)$$

$$T_{0x}' = T_x$$

$$T_{0y} = T_y + \frac{V_y^2}{2 c_p}$$

$$T_{0y}' = T_y \left[1 + \frac{k-1}{2} (M_y')^2\right]$$

$$P_{0x} = P_x \left(1 + \frac{k-1}{2} M_x^2\right)^{\frac{k}{k-1}}$$

$$P_{0x}' = P_x$$

$$P_{0y}' = P_y \left(1 + \frac{k-1}{2} M_y^2\right)^{\frac{k}{k-1}}$$

$$P_{0y}' = P_y \left[1 + \frac{k-1}{2} (M_y')^2\right]^{\frac{k}{k-1}}$$

Through the use of these relations and the boxed equations of note 12 the shock relations for a moving wave may be found. It is worthy of note that the change in stagnation temperature is dependent on the observer's motion, as indicated in the following expressions.

$$\Delta T_0 = T_{0y} - T_{0x} = T_y - T_x + \frac{V_y^2 - V_x^2}{2 c_p} = 0$$

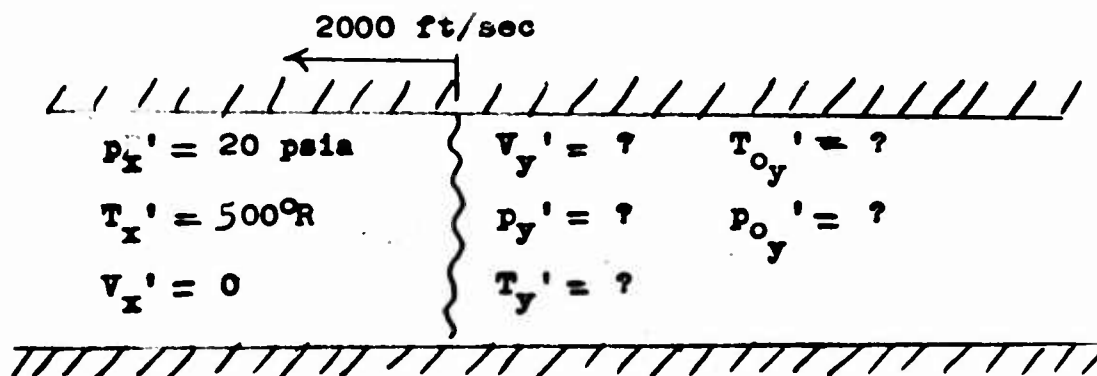
$$\Delta T_0' = T_{0y}' - T_{0x}' = T_y - T_x + \frac{(V_y - V_x)^2}{2 c_p}$$

and since  $\Delta T = 0$ , we find

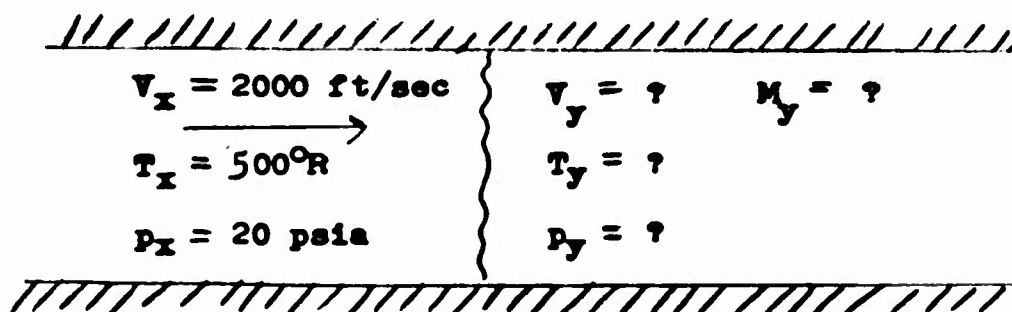
$$\Delta T_0' = \frac{V_x(V_x - V_y)}{c_p}$$

To illustrate the application of the foregoing consider finding the change of stream properties across a wave front propagating into a stationary gas with a wave velocity of 2000 feet per sec. Let the gas ahead of the wave be at 20 psia and 500°R.

We have then the following schematically.



To solve the problem we reduce it first to one for which the normal shock equation of note 12 apply. This is done schematically in the figure below by taking the point of view of one traveling with the wave.



The latter problem is easily solved and the properties behind the wave front advancing at 2000 ft/sec can then be found through the equations presented above. The solution to the problem proceeds as below.

We have

$$M_x = \frac{2000}{49.1 \sqrt{500}} = 1.82$$

From normal shock tables then

$$M_y = 0.612$$

$$\left( \frac{V_x}{V_y} \right) = \frac{\rho_y}{\rho_x} = 2.4 \quad \text{since} \quad \rho_x V_x = \rho_y V_y$$

$$\frac{p_y}{p_x} = 3.7; \quad \frac{T_y}{T_x} = 1.55$$

$$V_y = \frac{2000}{2.4} = 834 \text{ ft/sec.}$$

$$p_y = 3.7(20) = 74 \text{ psia}$$

$$T_y = 1.55(500) = 775^\circ\text{R}$$

Therefore, we find,

$$M_y' = M_y - \sqrt{\frac{T_x}{T_y}} \quad M_x = 0.612 - \sqrt{\frac{1}{1.55}} \quad 1.82$$

$$M_y' = -0.85 \text{ (minus sign means gas is moving to left)}$$

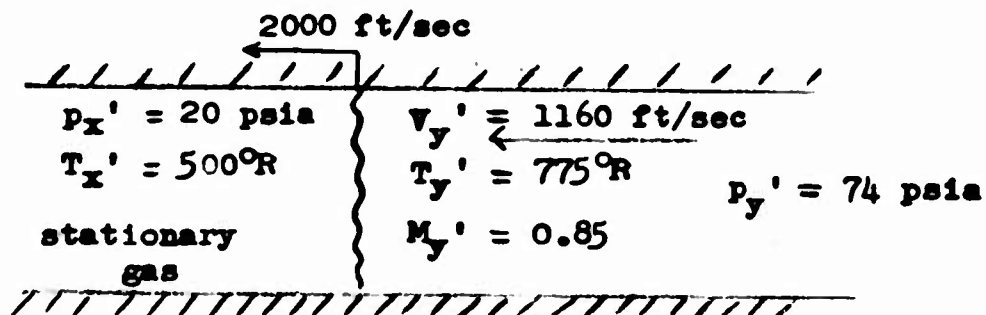
$$V_y' = M_y' 49.1 \sqrt{T_y'} = (0.85)(49.1) \sqrt{775} = 1160 \text{ ft/sec.}$$

$$T_{oy}' = \left(\frac{T_o}{T}\right)_y, \quad T_y' = \frac{775}{0.87} = 891^\circ\text{R}$$

$$p_{oy}' = \left(\frac{p_o}{p}\right)_y, \quad p_y' = \frac{74}{0.6235} = 118.9 \text{ psia}$$

Summarizing:

$V_y' = 1160 \text{ ft/sec}$	$T_y' = 775^\circ\text{R}$
$M_y' = 0.85$	$T_{oy}' = 891^\circ\text{R}$
$p_y' = 74 \text{ psia}$	$p_{oy}' = 118.9 \text{ psia}$



Notice that the gas following the blast wave is falling farther and farther behind the wave front since its velocity is less than the wave velocity.

## BIBLIOGRAPHY

### Lecture Note 2 - Conservation of Mass

Hunsaker and Rightmire, Engineering Applications of Fluid Mechanics, Chapter III, McGraw-Hill, 1947.

### Lecture Note 3 - Momentum Equation

Hunsaker and Rightmire, Engineering Applications of Fluid Mechanics, Chapter VI, Prandtl-Tritjens, Fundamentals of Hydro-and Aero-Mechanics, page 233 et seq., McGraw-Hill, 1934.

### Lecture Note 4 - First Law of Thermodynamics

Hunsaker and Rightmire, Engineering Applications of Fluid Mechanics, Chapter V.

### Lecture Note 6 - Velocity of Sound

Liepmann and Puckett, Aerodynamics of a Compressible Fluid, page 19 et seq., Wiley, 1947.

VonKarman, Supersonic Aerodynamics - Principles and Applications, Journal of the Aeronautical Sciences, vol. 14, no. 7, pp. 373-402, July, 1947.

### Lecture Note 7 - Total Pressure and Total Temperature

Keenan, Thermodynamics, Chapter III, Wiley, 1941

Breadwell and Harris, Some Engineering Fundamentals of the Turbojet, Turboprop, and Subsonic Ramjet, Chapter II, AAF TR 5514, July, 1946

### Lecture Note 8 - Nozzle Design

Kiefer and Stuart, Principles of Engineering Thermodynamics, Chapter XI, Wiley, 1930.

### Lecture Note 9 - Nozzle Operating Characteristics

Hunsaker and Rightmire, Engineering Application of Fluid Mechanics, Chapter IX, McGraw-Hill, 1947.

Lecture Note 10 - Simple Area Flow

Shapiro, Hawthorne, and Edelman, The Mechanics and Thermodynamics of Steady One-Dimensional Gas Flow, Section 4 of Handbook of Supersonic Aerodynamics, compiled by Applied Physics Laboratory, The John Hopkins University, Silver Spring, Md.

Keenan and Kaye, Gas Tables, pp. 208-210, Wiley, 1948

Lecture Note 11 - Compressibility Phenomena

Taylor and Maccoll, the Mechanics of Compressible Fluids, in Aerodynamic Theory, Vol III (edited by W. F. Durand), Springer, 1935.

Hunsaker and Rightmire, Engineering Applications of Fluid Mechanics, Chapter IX, McGraw-Hill, 1947.

Lecture Notes 12 and 13 - Normal and Oblique Shock Waves

Dailey and Wood, Computation Curves for Compressible Fluid Problems, Wiley, 1949

Keenan and Kaye, Gas Tables, pp. 211-213, Wiley 1948

Liepmann and Puckett, Introduction to the Aerodynamics of Compressible Fluids, Chapter 4, Wiley, 1947.

Ferri, Elements of Aerodynamics of Supersonic Flows, Chapter III, MacMillan, 1949

Taylor and Maccoll, The Mechanics of Compressible Fluids, in Aerodynamic Theory, Vol. III, (edited by W. F. Durnad), Springer, 1935

Keenan, Thermodynamics, pp 334-335, Wiley, 1941

Lecture Notes 14 and 15 - Simple Frictional Flow

Shapiro, Hawthorne, and Edelman, The Mechanics and Thermodynamics of Steady One-Dimensional Gas Flow, in Section 4 of Handbook of Supersonic Aerodynamics, compiled by Applied Physics Laboratory, The Johns Hopkins University, Silver Springs, Md.

Keenan and Kaye, Gas Tables, pp. 210-211, Wiley, 1948

Keenan, Thermodynamics, pp. 331-332, Wiley, 1941

Lecture Notes 16 and 17 - Simple Heating Effects

Shapiro, Hawthorne, and Edelman, The Mechanics and Thermodynamics of Steady One-Dimensional Gas Flow, in Section 4 of Handbook of Supersonic Aerodynamics, Compiled by Applied Physics Laboratory, The Johns Hopkins University, Silver Springs, Md.

Keenan and Kaye, Gas Tables, pp. 209-210, Wiley, 1948

Keenan, Thermodynamics, pp 335, Wiley, 1941

Lecture Note 18 - Supersonic Diffuser

Kantrowitz and Donaldson, Preliminary Investigation of Supersonic Diffusers, NACA Wartime Report L-713, May, 1945

Ferri, Elements of Aerodynamics of Supersonic Flows, Chapter 9, MacMillan, 1949

Lecture Note 19 - Steadily Moving Shock Waves

Shapiro, Hawthorne and Edelman, The Mechanics and Thermodynamics of Steady One-Dimensional Gas Flow, in Section 4 of Handbook of Supersonic Aerodynamics, compiled by Applied Physics Laboratory, The Johns Hopkins University, Silver Springs, Md.

**AIR FORCE INSTITUTE OF TECHNOLOGY**

**Home Problems  
Thermodynamics of Gas Flow  
ME 257**

**Maj. D. H. Daley**

HOME PROBLEMS

The problems will, in general, be numbered as follows: 1.1, 1.2, 1.3, . . . , 2.1, 2.2, . . . . 3.1, 3.2, . . . , where 2.3, for example, reads problem number 3 of the problem set accompanying lecture note 2. (For all problems involving air, assume air to be a perfect gas, unless otherwise noted, with  $k = 1.4$ ,  $R = 1715 \text{ ft}^2/\text{sec}^2 \text{ } ^\circ\text{R}$ , and  $c_p = 6000 \text{ ft}^2/\text{sec}^2 \text{ } ^\circ\text{R}$ )

- 1.1 Show that the path of a constant pressure process of a perfect gas on a  $T$ - $v$  diagram is a straight line. (Investigate slope  $dT/dv$  by differentiating equation of state).
- 1.2 The logarithmic differential of  $x$  is obtained by differentiating  $(\ln x)$  and is  $dx/x$ . Show that the logarithmic differentials of the properties  $p$ ,  $T$  and  $\rho$  ( $\rho = 1/v$ ) of a perfect gas are related by  $dp/p = dT/T + d\rho/\rho$ . (Hint: Write equation of state in logarithmic form.) Does  $d\rho/\rho = dv/v$ ? Prove.
- 1.3 The internal energy of 1.2 slugs of air in a rigid non-conducting container is increased as a paddle wheel in the container is turned by a mass of 20 slugs descending 200 ft. at a location where the acceleration of gravity is  $25 \text{ ft/sec}^2$ . Find
- $\Delta u$  (83,400 ft#/slug)
  - $\Delta h$  if initial temperature of air is  $70^\circ\text{F}$ .  
(117000 ft#/slug)
  - What would be the change of enthalpy of the system of gas if the acceleration of gravity is  $32.2 \text{ ft/sec}^2$ ?  
(151,000 ft#/slug)
- 1.4 Starting with the definition of entropy show that  $ds = c_p dT/T - R dp/p$ . Using this result and results of problem 1.2 along with  $c_p - c_v = R$  obtain,
- $ds = c_v \frac{dT}{T} - R \frac{d\rho}{\rho} = c_v \frac{dT}{T} + R \frac{dv}{v}$  ;  $s_2 - s_1 = c_v \ln \frac{T_2}{T_1} + R \ln \frac{v_2}{v_1}$
  - $ds = c_v \frac{dp}{p} - c_p \frac{dv}{v} = c_v \frac{dp}{p} + c_p \frac{dv}{v}$  ;  $s_2 - s_1 = \text{etc.}$
- 1.5 Dividing  $ds$  as given in (1.4) by  $c_p$  we obtain

$$\frac{ds}{c_p} = \frac{dT}{T} - \frac{R}{c_p} \frac{dp}{p} = \frac{dT}{T} - \frac{k-1}{k} \frac{dp}{p} . \text{ Thus for a reversible adiabatic process with } \frac{ds}{c_p} = 0 \text{ we have } \frac{dT}{T} = \frac{k-1}{k} \frac{dp}{p} \text{ or } \frac{T_2}{T_1} = \left( \frac{p_2}{p_1} \right)^{\frac{k-1}{k}}$$

Show similarly, using 1.4 (a) and (b), that



$$\frac{T_2}{T_1} = \left( \frac{v_1}{v_2} \right)^{k-1} = \left( \frac{\rho_2}{\rho_1} \right)^{k-1}$$

and

$$\frac{p_2}{p_1} = \left( \frac{\rho_2}{\rho_1} \right)^k = \left( \frac{v_1}{v_2} \right)^k$$

for an isentropic process.

- 1.6 Plot a temperature-entropy diagram for air with lines of constant pressure and specific volume thereon for  
 $p = 120, 60, 30, \text{ and } 15 \text{ psia}$   
 $v = 418 \text{ and } 317.5 \text{ ft}^3/\text{slug}$   
 Show an enthalpy scale along with the temperature scale. Select as a reference state of zero entropy and enthalpy air at 15 psia and 400°R. Let temperature and entropy scales range from 0°R to 1800°R (1° = 200°R) and

$$0 \frac{\text{ft lbf}}{\text{slug}^\circ\text{R}} \text{ to } 9600 \frac{\text{ft lbf}}{\text{slug}^\circ\text{R}} \quad (1^\circ = 1250 \frac{\text{ft lbf}}{\text{slug}^\circ\text{R}}) \text{ respectively.}$$

- 1.7 Graphically check the slopes  $\left( \frac{dT}{ds} \right)_v = \frac{T}{c_v}$  and  $\left( \frac{dT}{ds} \right)_p = \frac{T}{c_p}$  at any given

temperature on the diagram of problem 1.6.

- 1.8 Find: (1) The work done by, (2) the heat received by, the increase in (3) internal energy, (4) enthalpy and (5) entropy of a system of one slug of air which is initially at 120 psia and 1200°R as its specific volume increases to 418 ft<sup>3</sup>/slug by the following processes. (Sketch each process where possible on the T-s diagram of problem 1.6)

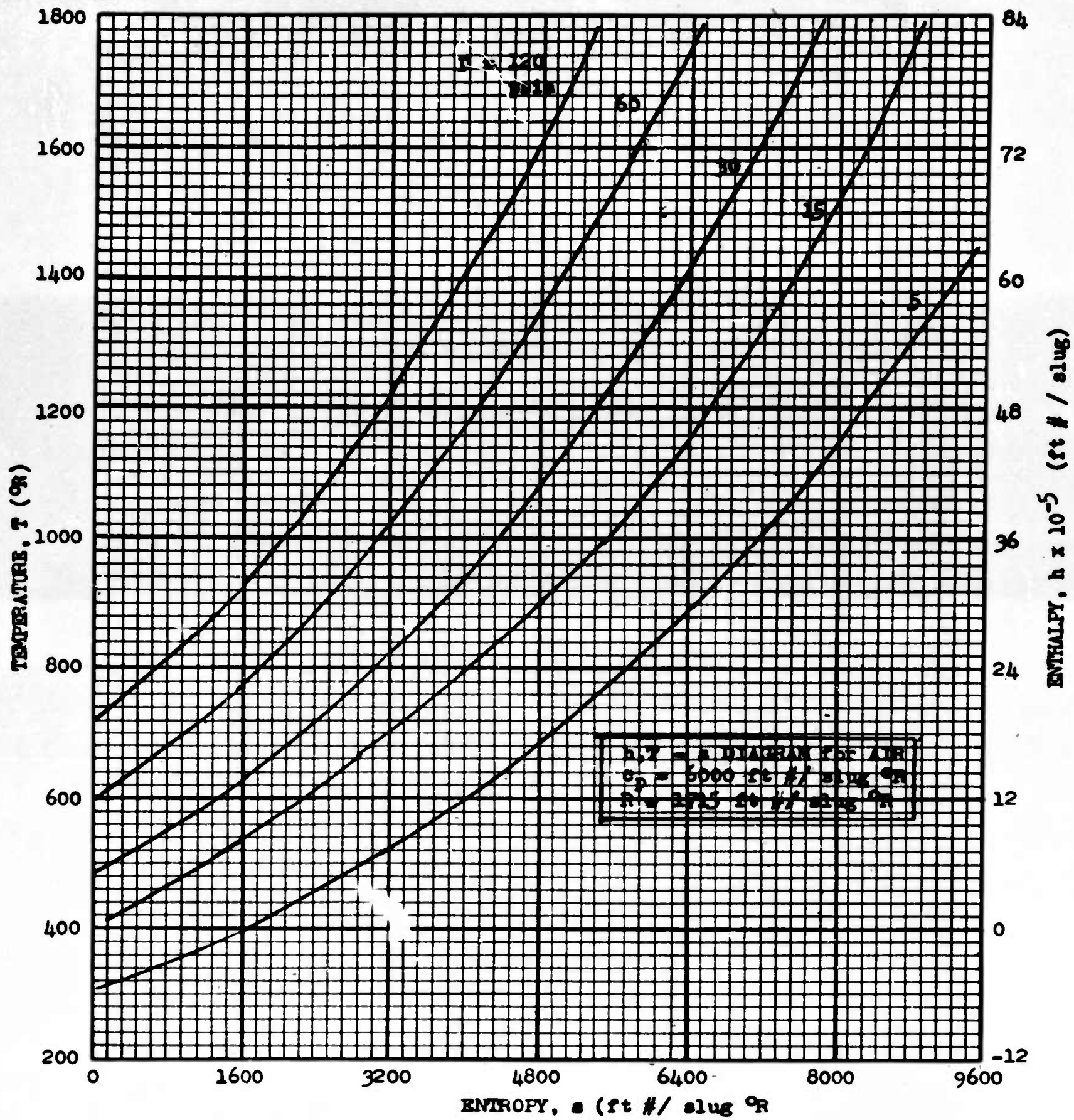
- (a) Reversible constant pressure
- (b) Reversible constant temperature
- (c) Reversible constant internal energy
- (d) Reversible constant enthalpy
- (e) Reversible adiabatic
- (f) Adiabatic expansion into an exhausted chamber.

- 1.9 Air expands through a nozzle from a large reservoir wherein  $T = 80^\circ\text{F}$ . What is the temperature of the air leaving the nozzle if the nozzle exit velocity is 1200 ft/sec? Would this temperature be measured by a stationary thermometer placed at the exit of the nozzle?

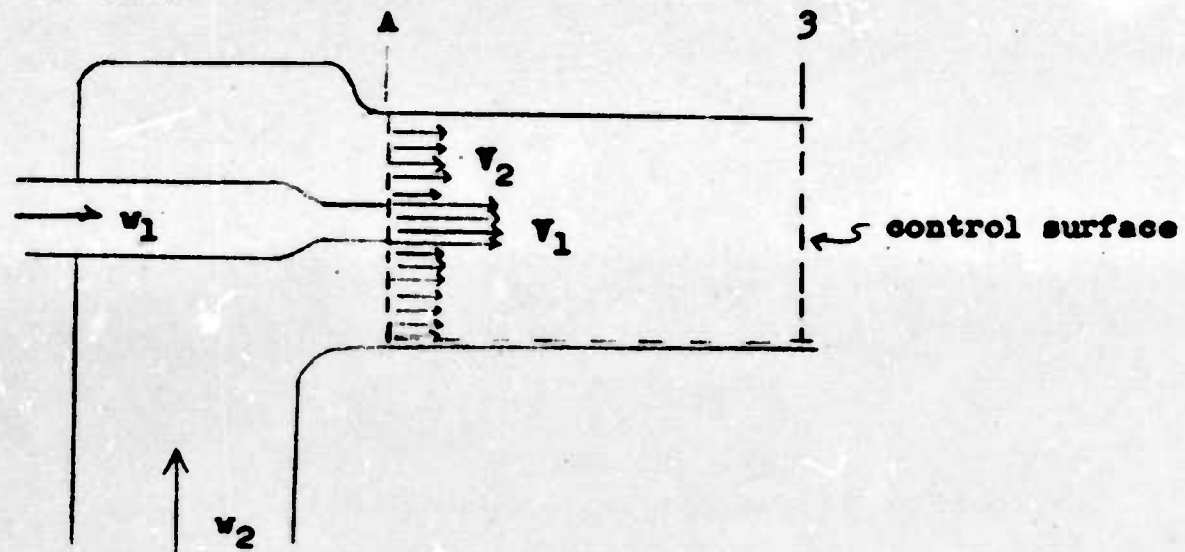
- 1.10 Air enters a reversible adiabatic turbine with  $p_1 = 60 \text{ psia}$ ,  $T_1 = 1800^\circ\text{R}$ ,  $V_1 = 200 \text{ ft/sec}$ , through an area  $A_1$  of 30 sq. inches. The air leaves the turbine with negligible velocity at 30 psia. Find the turbine horse-power output. (417 hp).

- 1.11 Solve for  $\Delta h$  of 1.10 graphically using  $h - s$  diagram of page 1.4.

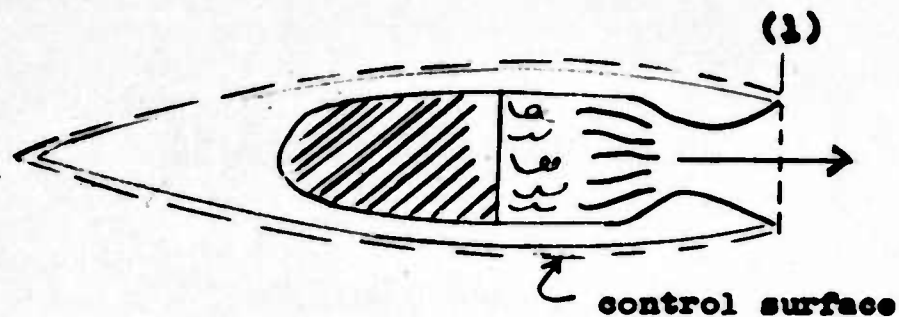
- 1.12 A body of 10 lbm mass is acted upon by a horizontal force of one pound at a location where the equivalent acceleration of gravity is  $20 \text{ ft/sec}^2$ . Neglecting friction, what is the horizontal acceleration due to the one pound force?
- 1.13 What would be the horizontal acceleration in Problem 1.12 if the body had a weight of 10 pounds at the location where equivalent  $g$  is  $20 \text{ ft/sec}^2$ ?



- 2.1 A jet pump is indicated schematically in the figure herewith. A high velocity jet of water flows out of the small nozzle with mass rate of flow  $w_1$ . This flow entrains water of mass flow  $w_2$  as shown. Using subscript 2 for stream properties of the entrained fluid at station A and subscript 1 for stream properties of the jet at section A write the continuity equation for the control volume indicated by dashed line.



- 2.2 Consider a rocket operating on a test stand. Apply the continuity equation to the control volume indicated.

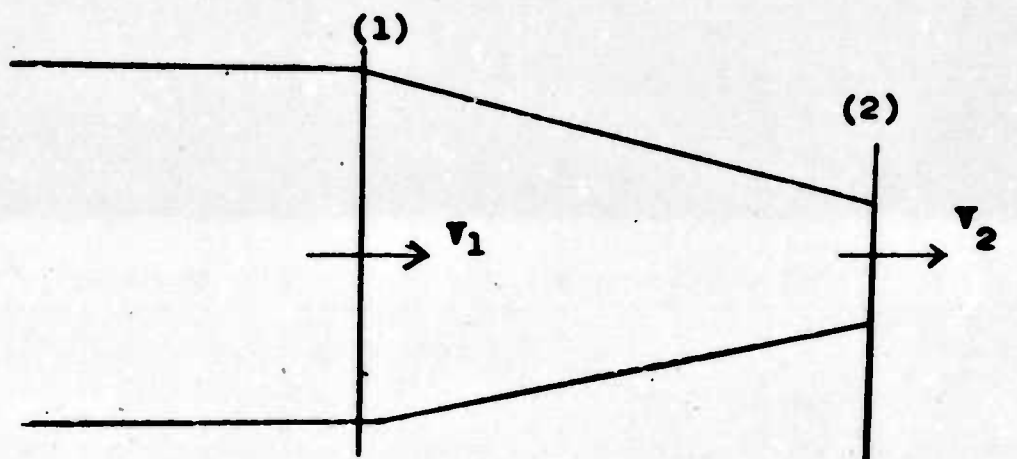


- 3.1 Apply the momentum equation to the pump of problem 2.1. Let the wall-fluid shearing stress be  $\tau_w$ . Obtain

$$p_1 A_1 + p_2 A_2 - p_3 (A_1 + A_2) - \int \tau_w dA_w = \rho (A_1 V_1 + A_2 V_2) - \rho A_1 V_1 - \rho A_2 V_2$$

- 3.2 Find the force of fluid on convergent portion of duct for the flow indicated below.

$$\rho_{H_2O} = \frac{62.4}{32.2} = 1.94 \frac{\text{slugs}}{\text{ft}^3}$$



$$p_1 = 20 \text{ psia}$$

$$A_1 = 2 \text{ ft}^2$$

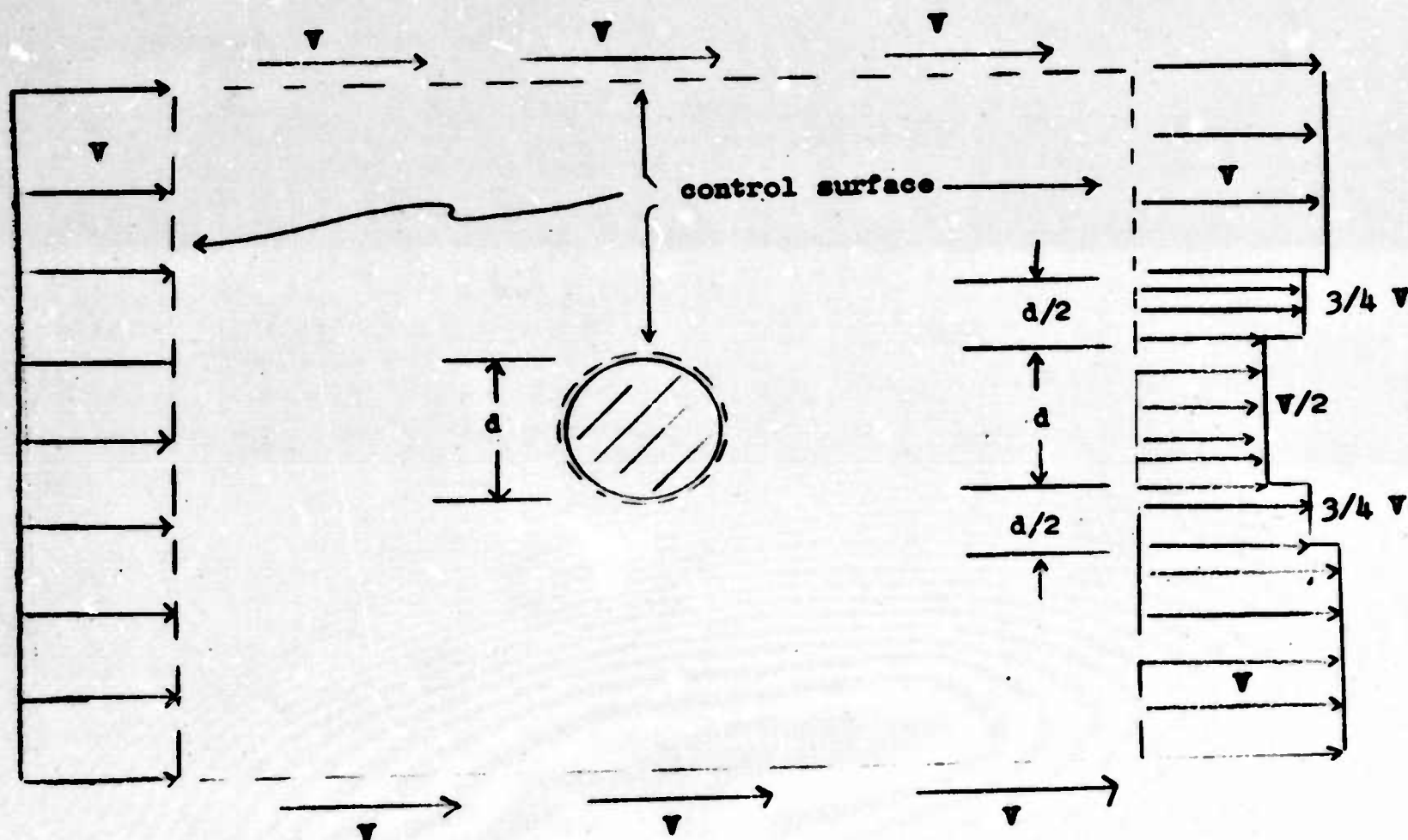
$$p_2 = 14.7 \text{ psia}$$

$$A_2 = 1 \text{ ft}^2$$

$$V_2 = 20 \text{ ft/sec}$$



- 3.3 Apply momentum equation to rocket of problem 2.2 and justify the assumption of  $\frac{dM}{dt} = 0$ . This assumption is used in the analysis of rocket thrust on a test stand using a solid propellant as indicated in sketch of problem 2.2.
- 3.4 (a) With the aid of a sketch show what forces are considered significant and are included in the term  $\vec{F}$ .
- (b) For each term in  $\vec{F}$  (of part (a)) give an example in which the term is negligible or zero.
- 3.5 In an experiment to determine drag, a circular cylinder of diameter  $d$  was immersed in a steady two-dimensional incompressible flow. Measurements of velocity and pressure were made at rectangular boundaries of the control surface shown. The pressure was found to be uniform over the rectangular portions of the control surface boundaries. The x-component of velocity at the control surface boundary was approximately as indicated in the sketch.



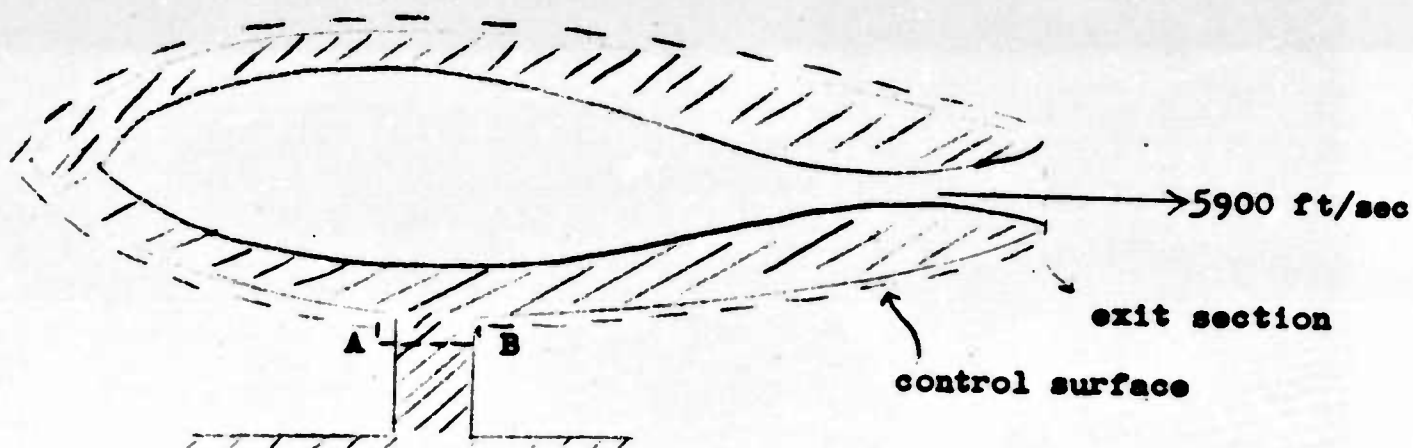
From the measured data calculate the drag coefficient of the cylinder defined as

$$C_D = \frac{\text{Drag force per unit length}}{(1/2 \rho V^2) d}$$

- 3.6 A rocket is mounted on a test stand as indicated below. The propellant consumption rate of the rocket is 5 lbm/sec. The rocket exhausts into a surrounding atmosphere of 10 psia with an exhaust jet velocity of 5900 ft/sec.

The thrust of the rocket is defined as the shear force acting in the support rod at the control surface section A-B.

- (a) Using the control surface shown develop an expression for the rocket thrust.
- (b) What is the magnitude of the thrust? (Treat the exhaust gases as perfect with the properties of air)
- (c) If the rocket nozzle exit section pressure is 10 psia and the surrounding atmosphere pressure were 12 psia, how would the thrust be affected? Explain briefly.



- 3.7 In a wind tunnel drag test of a missile model at high subsonic Mach Numbers the stream conditions preceeding the test section (at station 1 in figure below) are uniform across the tunnel at

$$p_1 = 20 \text{ psia}$$

$$T_1 = 461^\circ\text{R}$$

$$V_1 = 844 \text{ ft/sec}$$

With the missile mounted in the tunnel the stream conditions at station 2 are found to be (assume stream properties uniform at 2)

$$p_2 = 14 \text{ psia}$$

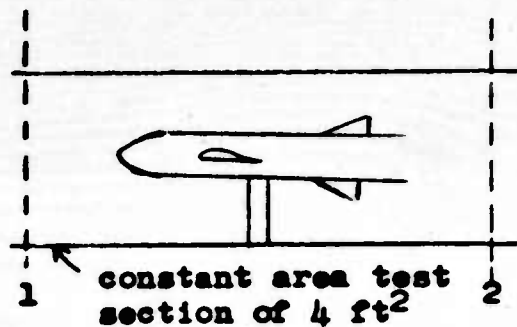
$$V_2 = 935 \text{ ft/sec.}$$

Without the missile in the tunnel the stream properties at station 2 are found to be

$$p_2 = 19.5 \text{ psia}$$

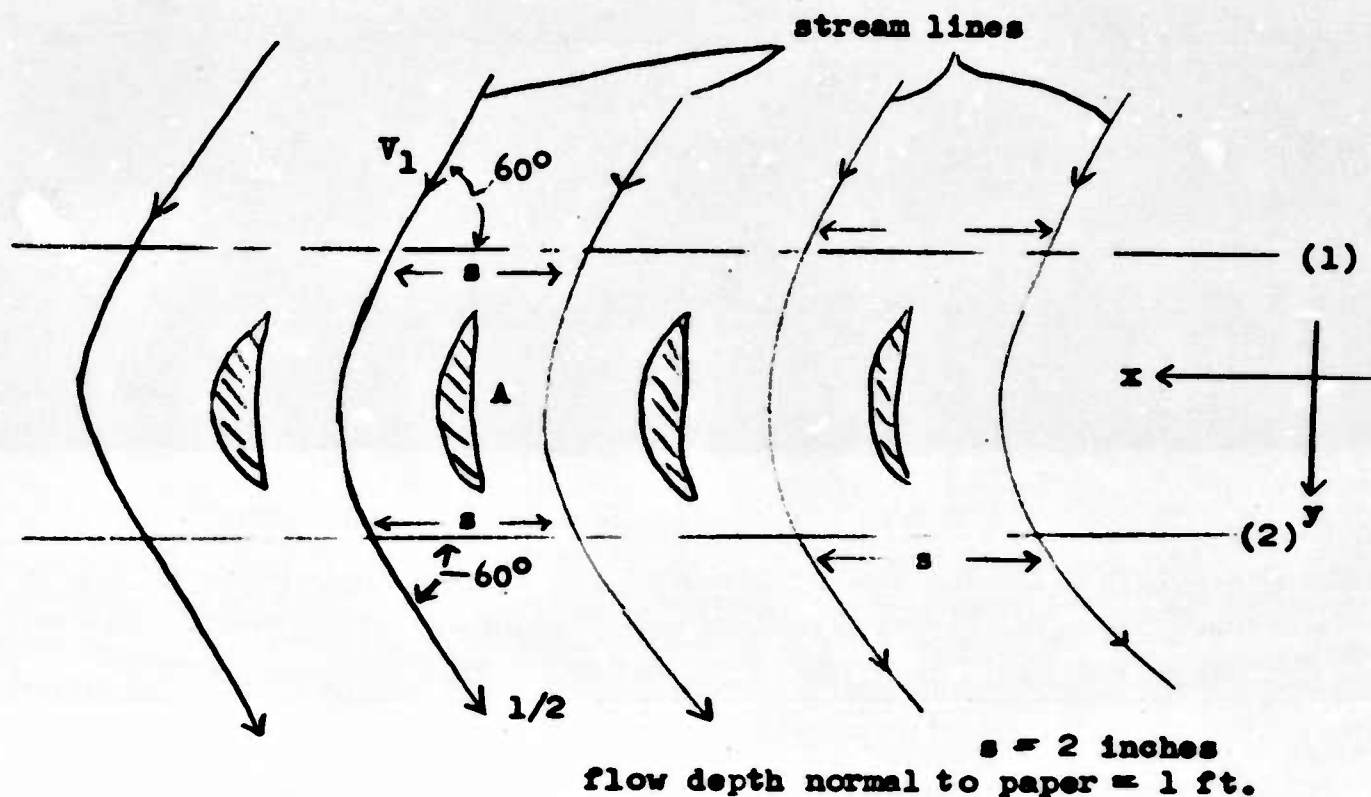
$$V_2 = 862 \text{ ft/sec.}$$

From these data estimate the missile drag coefficient based on the stream conditions at station 1. The model's wing area is 1.33 sq ft.





- 3.8 To obtain performance data on axial flow turbine blades, the blades may be arranged in a cascade (arrangement of blades in a manner similar to slats in a venetian blind) and tested in a two-dimensional wind tunnel as shown below.



In such a test the stream properties are uniform at section (1) and (2) and are

$$P_1 = 15 \text{ psia}$$

$$P_2 = 14.97 \text{ psia}$$

$$V_1 = 60 \text{ ft/sec}$$

Treating the air as incompressible ( $\rho = 0.078 \text{ lbm/ft}^3 = 0.00239 \text{ slug/ft}^3$ ) find the x and y components of the force acting on blade 'A' of the cascade

- 3.9 Consider the steady flow of an incompressible fluid in a constant area pipe of diameter  $D$ . Using the continuity and momentum equations show that the pressure drop between two stations, (1) and (2), a distance  $L$  apart is given by

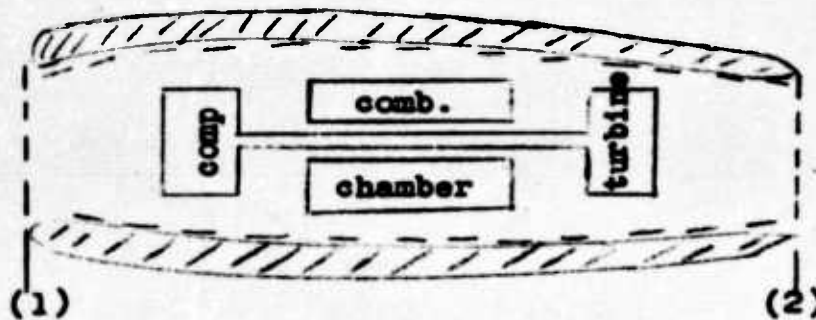
$$P_2 - P_1 = -f \frac{V^2}{2} \frac{L}{D}$$

where  $f$  is the pipe friction factor defined as

$$f = 4 \left( \frac{\tau_w}{\rho V^2 / 2} \right) \quad (\tau_w = \text{wall-fluid shearing stress})$$

(Note: Another friction factor sometimes used is  $f = \frac{\tau_w}{\rho V^2 / 2}$ )

- 3.10 The stream properties at the inlet and exit of a turbo-jet engine are given below. Determine the internal force of fluid on duct for these conditions.



$$A_1 = 2.37 \text{ ft}^2$$

$$P_1 = 20.3 \text{ psia}$$

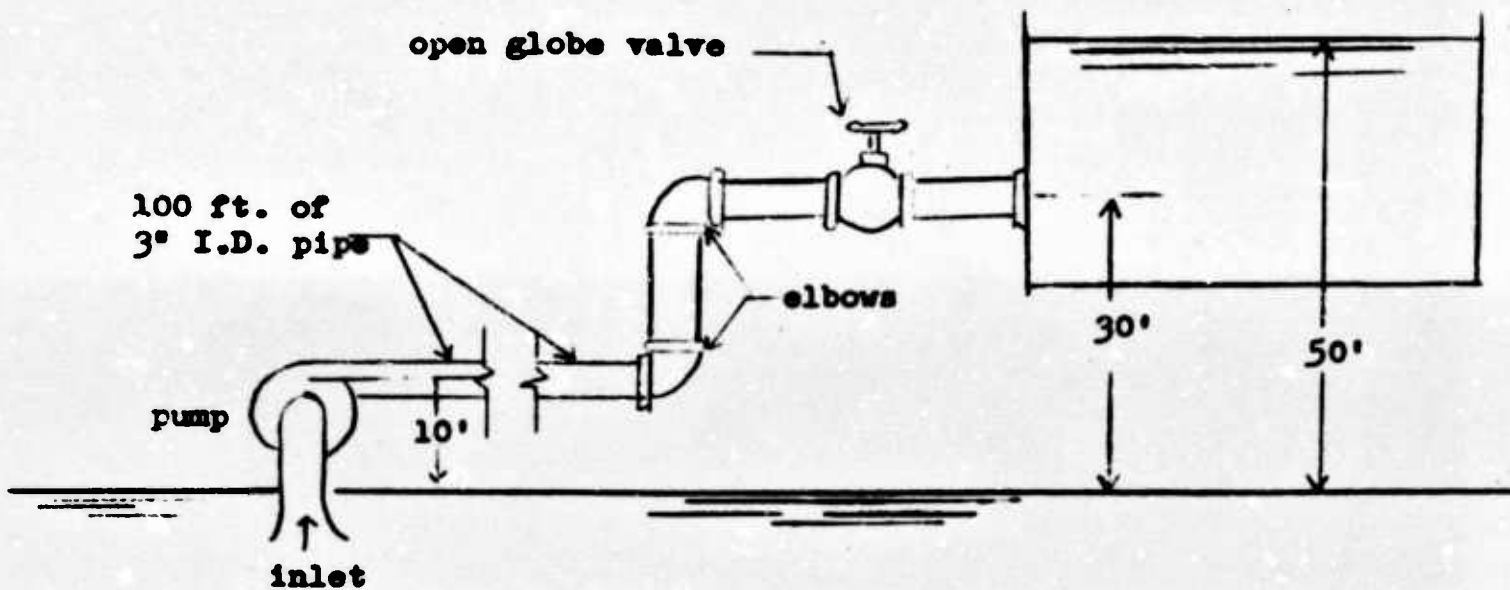
$$V_1 = 392 \text{ ft/sec}$$

$$T_1 = 570^\circ\text{R}$$

$$A_2 = 1.8 \text{ ft}^2$$

$$P_2 = 15.3 \text{ psia}$$

$$V_2 = 1980 \text{ ft/sec}$$



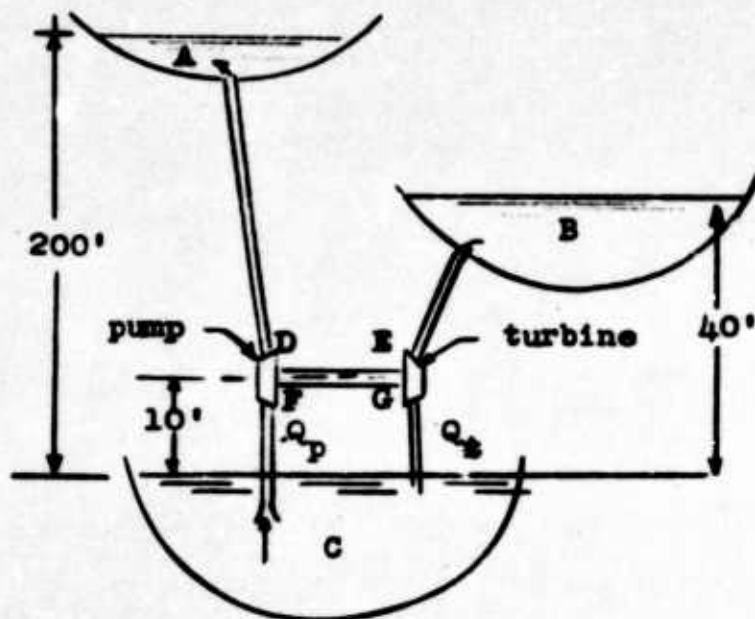
- 5.1 A pump discharges water through the system shown at the rate of 500 gallons/min. The loss coefficients are:

Device	C
Inlet	0.5
Elbow	1.0
Open Globe Valve	8.0
Sudden enlargement	as per theory
Pipe	$0.02 \frac{L}{D}$

The pump efficiency is 70%

- Estimate the gauge pressure at the pump discharge, in psig.
- Estimate the power required by the pump, in horsepower.

5.2



In certain regions where hydro-electric power is available, a large amount of water at low head is available in reservoir B. This amount is more than ample to take care of power demands for most of the year, but at certain periods of the year, there is a shortage. In such cases, if a small basin A is available near by at a very high head, it may be economical to pump water to A during the times when there is excess capacity in B, thus making the water in A available when B is low in water. The sketch shows such a system for transferring water from B to a, with a direct drive between the turbine and pump.

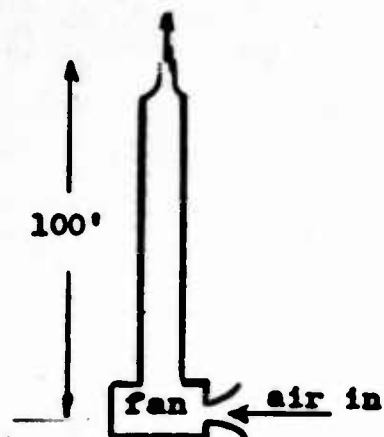
Determine the ratio  $Q_P/Q_T$  of water pumped to A to total water used by the turbine, assuming,

$$\eta_P = \eta_T = 0.80$$

$$H_{l_{BE}} = 4' ; H_{l_{DA}} = 20' ; H_{l_{BC}} = H_{l_{CA}} = 2'$$

$$(\text{ANS: } Q_P/Q_T = 0.123)$$

5.3



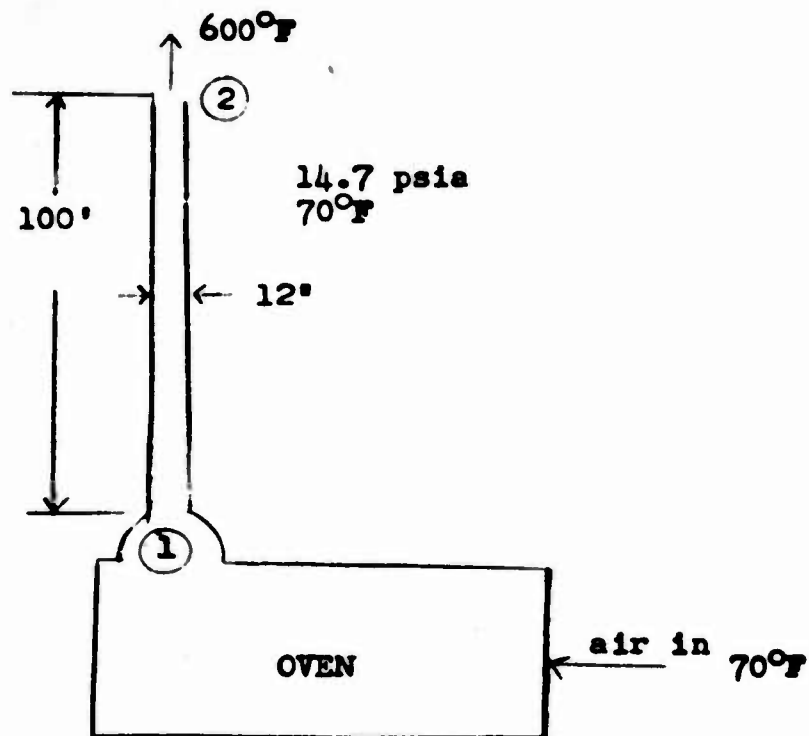
Air is drawn into a fan at standard conditions ( $\rho = 0.075 \text{ lbm/ft}^3$ ) and is discharged from a nozzle placed at the exit of a 100-ft stack. The nozzle exit diameter is 1 ft, and the air flow is  $100 \text{ ft}^3/\text{sec}$ . Assuming that all losses in the system are negligible except for the losses in the fan, and that the latter has an efficiency of 60%, estimate

- the power input to the shaft of the fan, in horsepower
- the lost head in the fan, in ft. of air.

$$\text{ANS: (a) } 5.7 \text{ hp}$$

$$(b) 167 \text{ ft}$$

5.4



Air is heated to 600°F in an oven and is then exhausted to the atmosphere through a 12-inch sheet metal stack 100 ft. high.

Assume that in addition to pipe friction there are miscellaneous losses amounting to two kinetic energy heads (based on the velocity in the stack), and that the static pressure at (1) is the same as the static pressure at the air inlet.

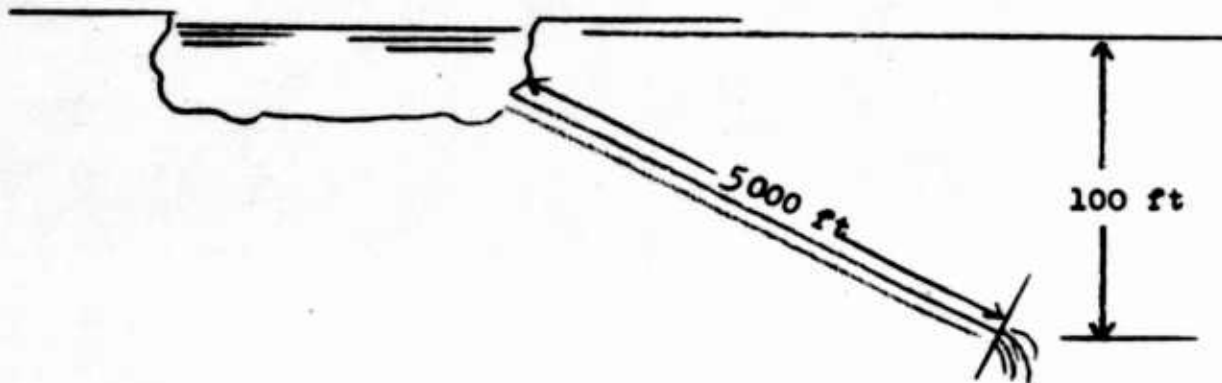
Estimate the mass rate of flow of air through the system, in lbm/sec.

The head loss in the pipe may be estimated from

$$H_{l_{\text{pipe}}} \cong 0.02 \frac{L}{D} \frac{V^2}{2g}$$

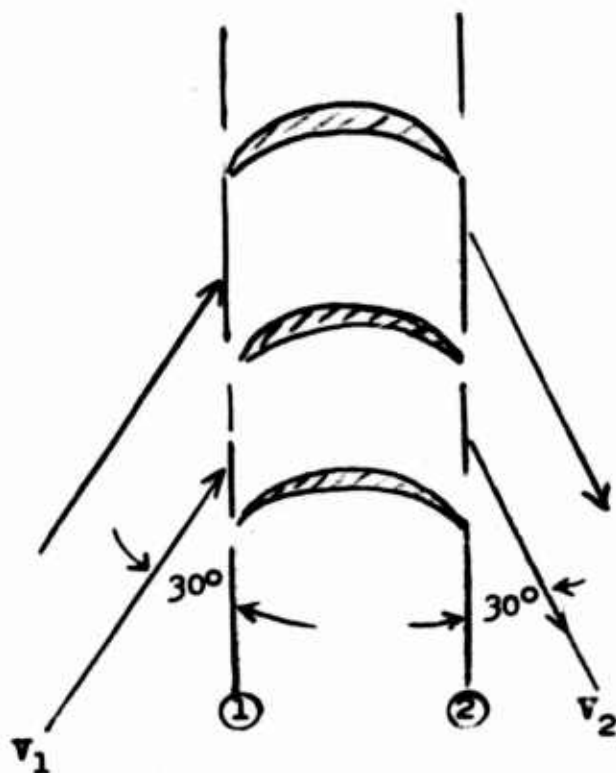
Answer: 1.05 lbm/sec

- 5.5 Water flows from a lake through a pipe of 1 foot inside diameter and 5000 feet long, discharging into the atmosphere at a point 100 feet below the lake's surface. If the friction factor inside the pipe is 0.02 what mass rate of flow would you expect to be discharged from the pipe? Neglect any head losses except those in the 5000 feet of pipe.



- 5.6 Estimate the volume of room air flow per minute through a household fireplace with a mean chimney gas temperature of  $860^{\circ}\text{F}$ . The chimney height is fifty feet. The flow losses in the chimney amount to two kinetic heads. Use chimney area of one  $\text{ft}^2$ .
- 5.7 In evaluating a particular chimney design for a furnace, measurements of the leaving gas velocity from the chimney and of the flue gas temperature are made. In a 100 ft. chimney of 12" diameter the flue gas temperature is found to be  $600^{\circ}\text{F}$  when the exit velocity from the chimney is 40 feet per second. If the ambient temperature is  $70^{\circ}\text{F}$  what is the magnitude of the flow losses through the chimney? (Assume the flue gases to be air)

- 5.8 Air enters the turbine cascade depicted below at 900 ft per second and at a pressure of 14.7 psia. The cascade is a constant passage area-impulse type. Treating the flow as incompressible and assuming a loss coefficient through the cascade of 0.2 find the axial and tangential force on the cascade per unit of flow passage area. The entering and leaving velocities make an angle of  $30^\circ$  with the plane of the turbine wheel. ( $\rho = 0.002378$  slugs/ft<sup>3</sup>)





- 5.9 Using the Hagen Poiseuille Law and the momentum equation applied to a control volume of radius  $R$  (pipe radius) show that for fully developed laminar pipe flow of an incompressible fluid the pipe friction factor is given by

$$f = \frac{64}{Re_D}$$

where  $Re_D = \frac{\rho V D}{\mu}$  and  $V$  = Bulk mean velocity

- 5.10 In an experiment to measure the viscosity of an oil, the oil flows through a 0.811 inch diameter pipe at a measured flow rate of 0.46 lbm/sec. The pressure of the oil measured at two points 2 ft apart is found to be

$$p_1 = 1.7 \text{ ft. oil}$$

$$p_2 = 1.6 \text{ ft. oil}$$

By observing the flow exit conditions from the pipe it is known that laminar flow exists. From the given data determine the dynamics viscosity of the oil. (oil sp. gr. is 0.8) What is the kinematic viscosity of the oil?

$$(\mu = 1.46 \times 10^{-4} \frac{\text{lb-f-sec}}{\text{ft}^2} \text{ or } \frac{\text{slug}}{\text{ft-sec}} \cdot 4.7 \times 10^{-3} \frac{\text{lbm}}{\text{ft-sec}}, 7 \text{ centipoise})$$

$$(\frac{\mu}{\rho} = 0.942 \frac{\text{ft}^2}{\text{sec}}, 8.75 \frac{\text{cm}^2}{\text{sec}} \text{ or stokes})$$

$$\text{Note: } 1 \text{ poise} \stackrel{d}{=} \frac{1 \text{ dyne-sec}}{\text{cm}^2}$$

$$1 \text{ dyne} = 2.248 \times 10^{-6} \text{ lbf}$$

$$1 \text{ cm} = 2.54 \text{ in.}$$

$$1 \text{ stoke} \stackrel{d}{=} 1 \frac{\text{cm}^2}{\text{sec}}$$

$$\text{For water at } 20^\circ\text{C}, \frac{\mu}{\rho} = 1 \text{ centipoise.}$$

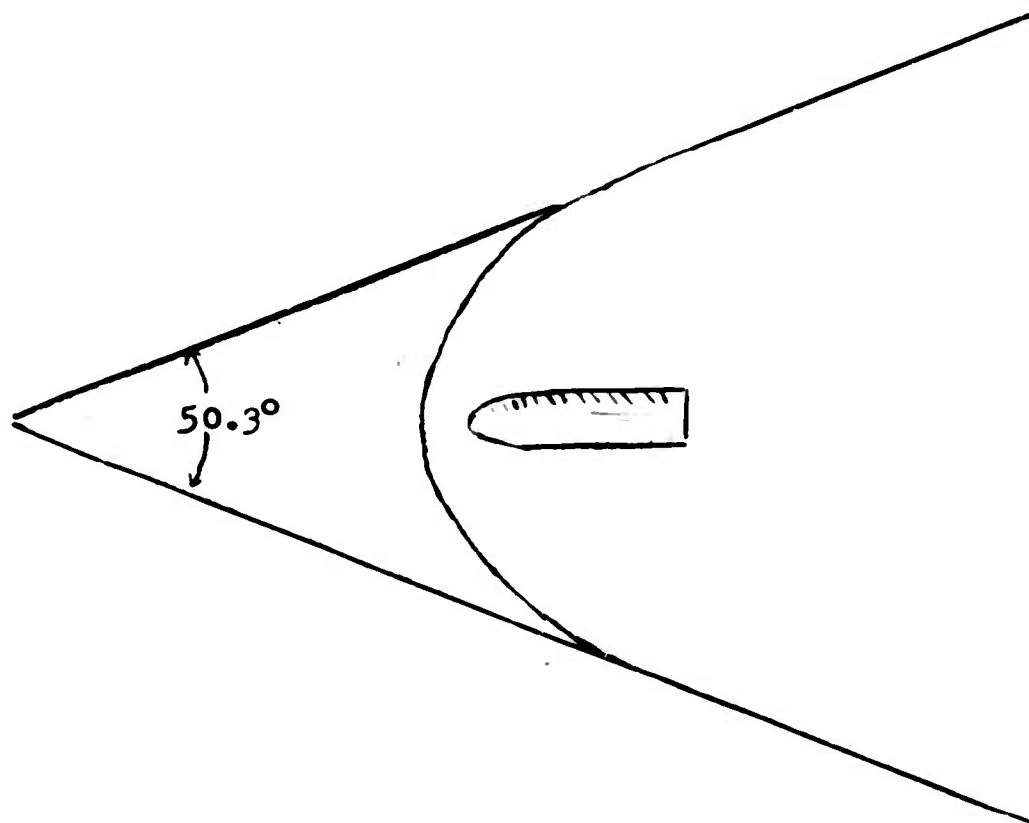


- 6.1 A projectile in flight carries with it a more-or-less conical-shaped shock front. From physical reasoning it appears that at great distances from the projectile this shock wave becomes truly conical and changes in velocity and density across the shock become vanishingly small.

Photographs of a bullet in flight show that at a great distance from the bullet the total included angle of the cone is  $50.3^\circ$ . The pressure and temperature of the undisturbed air are 14.62 psia and  $73^\circ\text{F}$ , respectively.

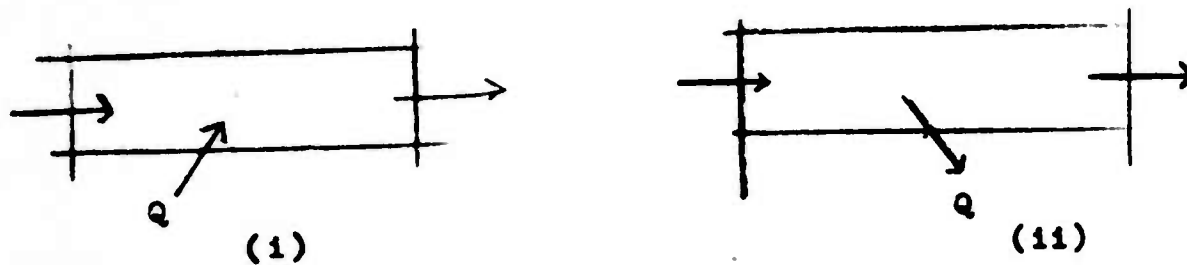
Calculate the velocity of the bullet, in ft/sec., and the Mach Number of the bullet relative to the undisturbed air.

(ANS.: 2680 ft/sec; 2.36)

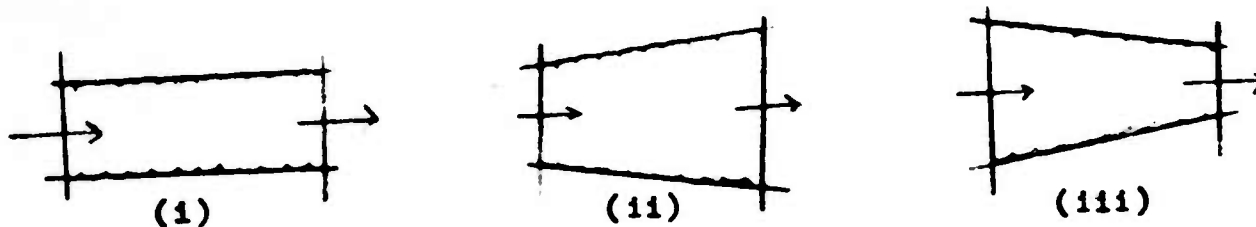


7.1 Give the value of  $dT_0$  and  $dp_0$  relative to zero for flow through the following ducts.

- (a) frictionless, diabatic (non-adiabatic), and constant area duct,  
(i) heating, (ii) cooling



- (b) Frictional, adiabatic, and (i) constant area, (ii) diverging area  
(iii) converging area.



- (c) frictionless, adiabatic, and (i) constant area (ii) diverging area and (iii) converging area.

7.2 How many independent properties are required to fix the state of a gas:

- (a) in the absence of motion, gravity, electricity, capillarity, and magnetism?  
(b) in the absence of gravity, electricity, capillarity, and magnetism?

7.3 In the case of (7.2b) above, will

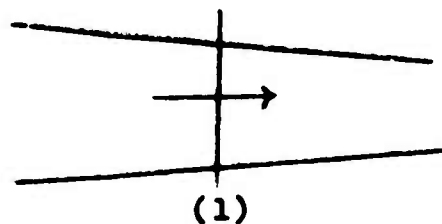
- (a)  $p$ ,  $T$ , and  $v$
- (b)  $p$ ,  $T$ , and  $p_0$
- (c)  $T$ ,  $T_0$ , and  $p$
- (d)  $p$ ,  $T$ , and  $V$
- (e)  $p$ ,  $p_0$ , and  $V$
- (f)  $T$ ,  $T_0$ , and  $V$
- (g)  $p_0$ ,  $T_0$ , and  $p$
- (h)  $p_0$ ,  $T_0$ , and  $T$
- (i)  $p_0$ ,  $T_0$ , and  $V$

fix the state of the gas?

7.4 Put the steady flow energy equation for flow of perfect gas with  $W_x = Q = 0$  in terms of Mach number starting with

$$h_1 + \frac{v_1^2}{2} = h_2 + \frac{v_2^2}{2}$$

7.5 Air is flowing through the passage indicated with the quantities given measured at section (1) where duct cross sectional area is  $0.2 \text{ ft}^2$ . Find the gas mass rate of flow.



$$T_{o1} = 734^\circ\text{R}$$

$$p_{o1} = 100 \text{ psia}$$

$$p_1 = 84.3 \text{ psia.}$$

7.6 For air

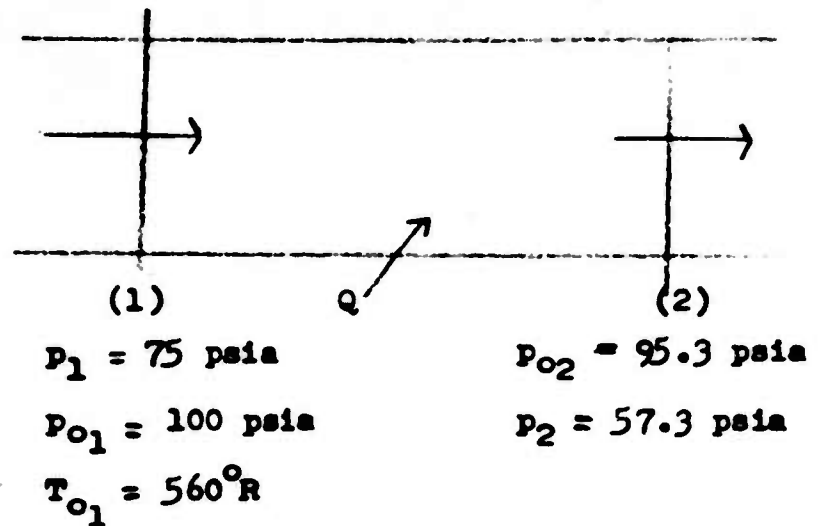
$$\left(\frac{p}{p_0}\right)_1 < m \text{ means } M_1 > 1.$$

$$\left(\frac{T}{T_0}\right)_1 < n \text{ means } M_1 > 1.$$

Determine  $m$  and  $n$

7.7 Air is flowing through a frictionless constant area duct with 600 Btu of heat added per slug of air between stations (1) and (2). For the conditions given below determine

## 7.7 (Cont.)

(a)  $T_1$ (b)  $V_1$ (c)  $M_1$ (d)  $T_{o2}$ (e)  $T_2$ (f)  $V_2$ (g)  $M_2$ 

$$Q = (600 \times 778) \frac{\text{ft. lbf}}{\text{slug}}$$

Compare stream properties at (1) with those at (2). Notice that pressure drops, Mach number increases, Velocity increases, etc.

A circular nozzle having the area distribution given in the accompanying chart is to expand air reversibly and adiabatically from a region in which

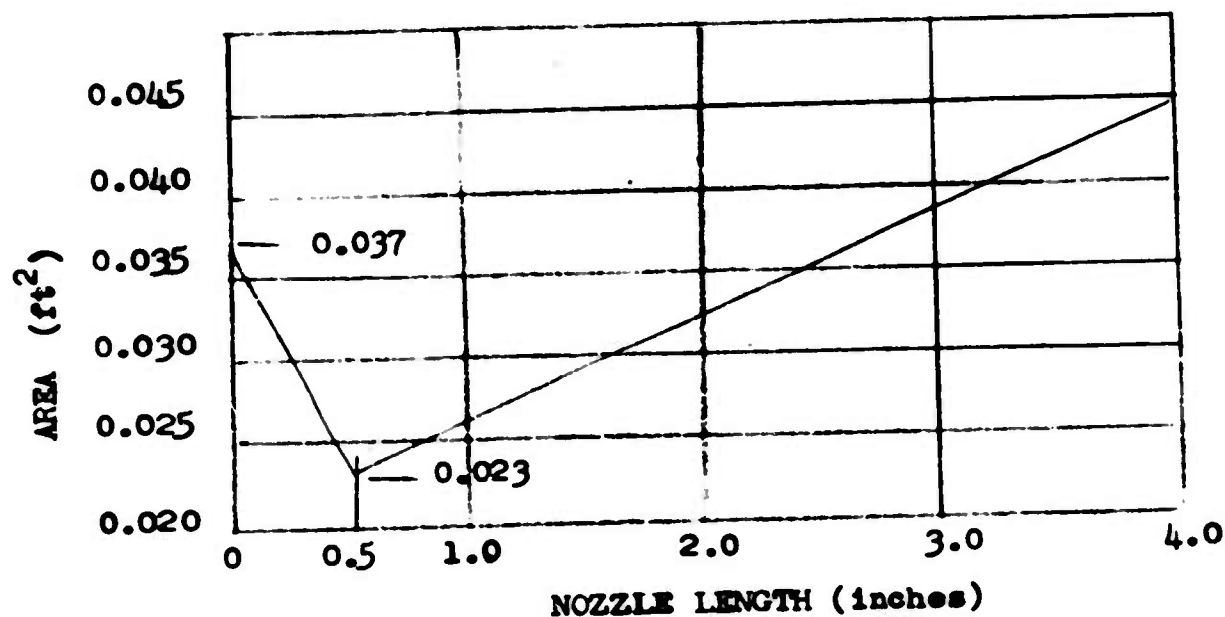
$$p_0 = 300 \text{ psia}$$

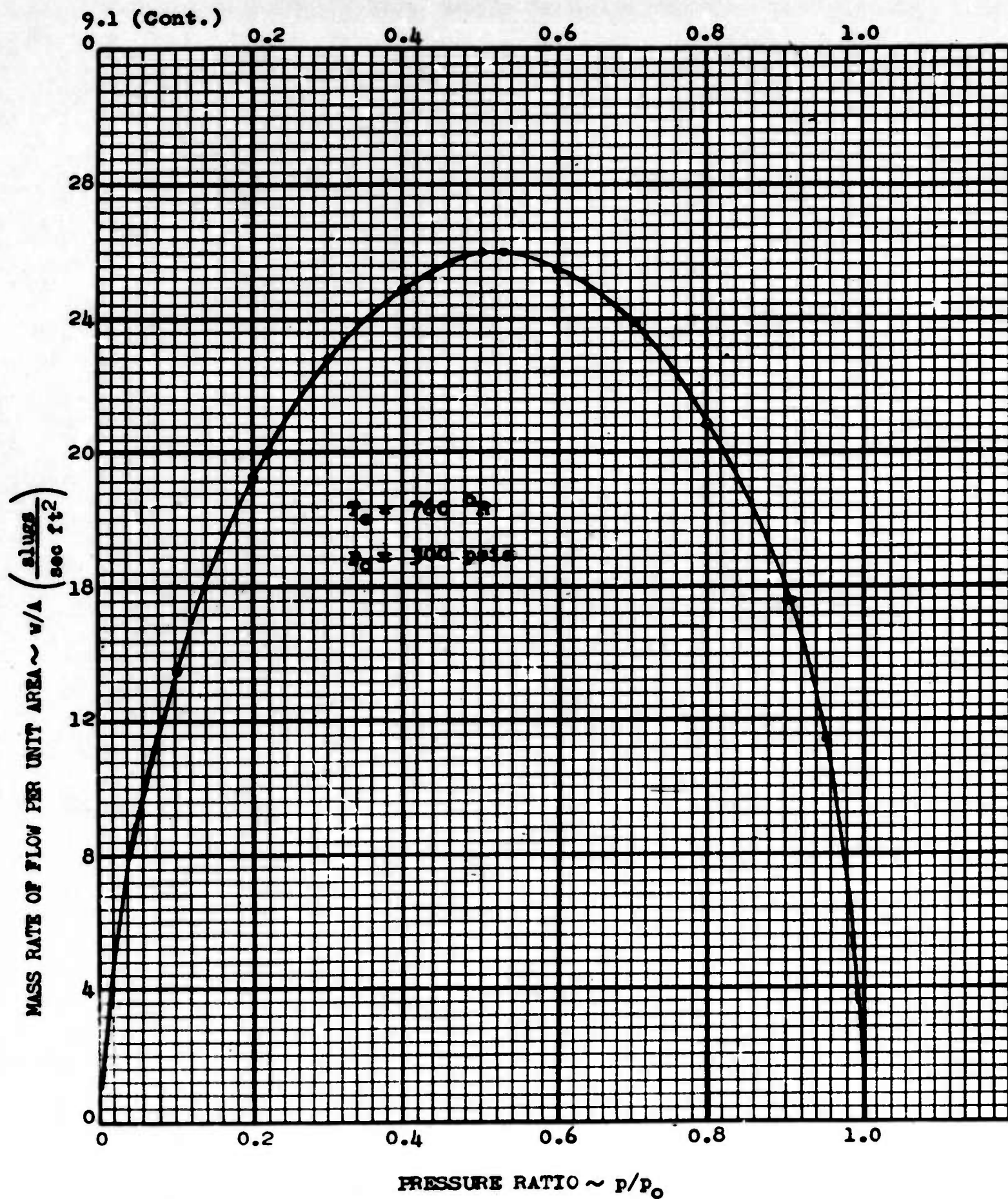
$$p = 265 \text{ psia}$$

$$T_0 = 760^\circ\text{R}$$

(See plot on next page.)

- Assuming supersonic flow downstream of the throat what is the exhaust region pressure,  $p_g$ , for reversible flow throughout? (Hint: Compute  $w$ ).
- Plot the pressure ratio  $p/p_0$  versus nozzle length for the expansion to  $p_g$  and sketch the process on the appropriate T - S diagram of the area distribution chart.
- Determine the exhaust region pressure,  $p_h$ , which gives subsonic flow in the diffuser section for maximum mass rate of flow.
- Plot  $(p/p_0)$  along the nozzle axis for the flow of part c and sketch the process on the T-S diagram for exhaust to  $p_h$ .
- Calculate the mass rate of flow when the exhaust region pressure is  $p_1 = 285 \text{ psia}$  and plot the pressure distribution for this case. Sketch process on appropriate T-S diagram. (Note that  $p_{\text{inlet}} > 265 \text{ psia}$  for this flow.)
- Plot the pressure distribution when the mass flow is 0.5 slugs per sec.





## Functions of Mach Number

Subsonic				Supersonic			
M	$\frac{W}{\sqrt{T_0}} \frac{1}{AP}$	M	$\frac{W}{\sqrt{T_0}} \frac{1}{AP}$	M	$\frac{W}{\sqrt{T_0}} \frac{1}{AP}$	M	$\frac{W}{\sqrt{T_0}} \frac{1}{AP}$
0.00	.0000	0.50	.4698	1.00	1.004	1.50	1.654
.01	.0092	.51	.4797	1.01	1.016	1.51	1.668
.02	.0184	.52	.4895	1.02	1.027	1.52	1.682
.03	.0275	.53	.4995	1.03	1.039	1.53	1.697
.04	.0367	.54	.5094	1.04	1.051	1.54	1.712
.05	.0459	.55	.5194	1.05	1.063	1.55	1.726
.06	.0551	.56	.5294	1.06	1.075	1.56	1.741
.07	.0643	.57	.5394	1.07	1.087	1.57	1.756
.08	.0743	.58	.5494	1.08	1.099	1.58	1.771
.09	.0826	.59	.5595	1.09	1.101	1.59	1.786
.10	.0916	.60	.5696	1.10	1.123	1.60	1.801
.11	.1010	.61	.5797	1.11	1.135	1.61	1.816
.12	.1102	.62	.5899	1.12	1.148	1.62	1.831
.13	.1195	.63	.6001	1.13	1.160	1.63	1.846
.14	.1287	.64	.6103	1.14	1.172	1.64	1.862
.15	.1379	.65	.6206	1.15	1.185	1.65	1.877
.16	.1471	.66	.6309	1.16	1.197	1.66	1.892
.17	.1564	.67	.6412	1.17	1.210	1.67	1.908
.18	.1656	.68	.6516	1.18	1.212	1.68	1.923
.19	.1749	.69	.6620	1.19	1.235	1.69	1.939
.20	.1842	.70	.6724	1.20	1.247	1.70	1.954
.21	.1934	.71	.6829	1.21	1.260	1.71	1.970
.22	.2027	.72	.6934	1.22	1.273	1.72	1.986
.23	.2120	.73	.7040	1.23	1.286	1.73	2.002
.24	.2213	.74	.7145	1.24	1.299	1.74	2.018
.25	.2307	.75	.7252	1.25	1.312	1.75	2.034
.26	.2400	.76	.7358	1.26	1.325	1.76	2.050
.27	.2494	.77	.7465	1.27	1.338	1.77	2.066
.28	.2588	.78	.7572	1.28	1.351	1.78	2.082
.29	.2682	.79	.7680	1.29	1.364	1.79	2.098
.30	.2776	.80	.7788	1.30	1.377	1.80	2.114
.31	.2870	.81	.7897	1.31	1.390	1.81	2.131
.32	.2965	.82	.8006	1.32	1.404	1.82	2.147
.33	.3059	.83	.8115	1.33	1.417	1.83	2.164
.34	.3154	.84	.8225	1.34	1.431	1.84	2.180
.35	.3249	.85	.8335	1.35	1.444	1.85	2.197
.36	.3344	.86	.8445	1.36	1.458	1.86	2.214
.37	.3440	.87	.8556	1.37	1.471	1.87	2.230
.38	.3535	.88	.8667	1.38	1.485	1.88	2.247
.39	.3631	.89	.8779	1.39	1.499	1.89	2.264
.40	.3727	.90	.8891	1.40	1.512	1.90	2.281
.41	.3823	.91	.9004	1.41	1.526	1.91	2.298
.42	.3919	.92	.9117	1.42	1.540	1.92	2.315
.43	.4016	.93	.9231	1.43	1.554	1.93	2.332
.44	.4112	.94	.9344	1.44	1.568	1.94	2.350
.45	.4210	.95	.9459	1.45	1.582	1.95	2.367
.46	.4307	.96	.9574	1.46	1.596	1.96	2.384
.47	.4404	.97	.9689	1.47	1.611	1.97	2.402
.48	.4502	.98	.9805	1.48	1.625	1.98	2.419
.49	.4600	.99	.9921	1.49	1.639	1.99	2.437
.50	.4698	1.00	1.0038	1.50	1.654	2.00	2.454

- 10.1 Starting with  $a^2 = kRT$  and using relations developed in note 10 show that

$$\frac{da}{a} = \frac{\frac{k-1}{2} M^2}{1-M^2} \frac{dA}{A}$$

- 10.2 Show that in subsonic isentropic flow

$$\frac{dp}{dA} > 0 \quad \frac{dM}{dA} < 0$$

$$\frac{dV}{dA} < 0 \quad \frac{dT}{dA} > 0$$

and vice versa for supersonic flow.

- 10.3 Derive

$$(a) \quad \frac{a^2}{k-1} + \frac{V^2}{2} = a_*^2 \frac{k+1}{2(k-1)}$$

using energy equation, equation for velocity of sound, and definition of starred quantities. Show that (a) reduces to

$$(b) \quad a_0^2 = \frac{k+1}{2} a_*^2$$

- 10.4 A supersonic wind tunnel is to be designed for a test section Mach number of 1.5. The inlet conditions to the tunnel are to be

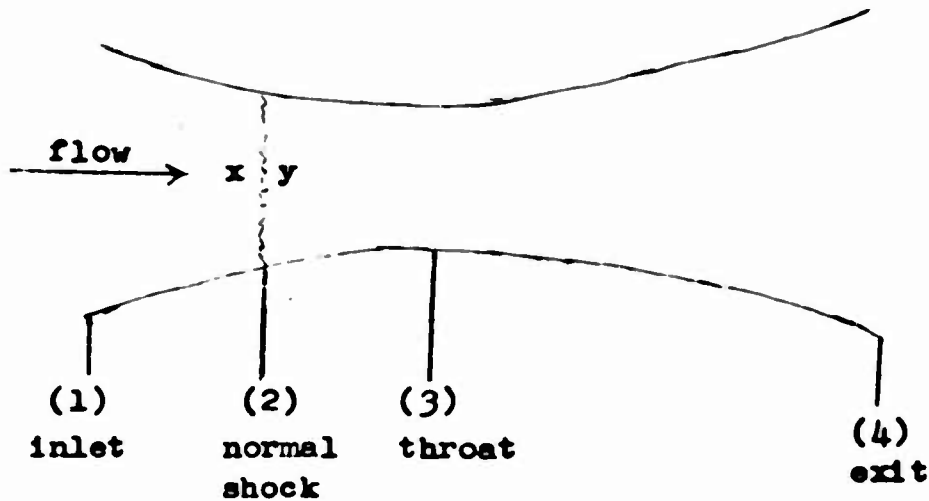
$$p_1 = 10.6 \text{ psia} \quad A_1 = 1.2 \text{ ft}^2$$

$$p_0 = 14.7 \text{ psia} \quad T_0 = 600^\circ \text{ R}$$

- (a) What is the required test section area?
- (b) Find  $T$ ,  $p$ ,  $V$  and  $w$  in the test section
- (c) If the tunnel exit area is  $0.2 \text{ ft}^2$  greater than the test section area what is the limiting exhaust region pressure for sonic throat conditions with  $p_0 = 14.7$  and  $T_0 = 600^\circ \text{ R}$ ?
- (d) Will the test section conditions correspond to those of part (b) up to the exhaust region pressure of part (a)?



- 12.1 Draw the pressure distribution through the nozzle of problem 9.1 assuming a normal shock occurs at the station corresponding to  $(p/p_0) = 0.2$  with reversible flow elsewhere. What exit region pressure is required to produce this flow?
- 12.2 A supersonic diffuser is operating with a supersonic inlet stream velocity and with a normal shock in the diffuser at station (2) as shown below. The stream velocity is reduced to a negligible magnitude at the exit of the diffuser.



Sketch the flow process through the diffuser on a T-s diagram indicating on the temperature axis

$$(a) \quad T_{0x}, T^*_x, T_x, T_{0y}, T^*_y, T_y.$$

Also sketch in pressure lines corresponding to

$$(b) \quad p_{0x}, p^*_x, p_{0y}, p^*_y.$$

Make the sketch sufficiently large to be clear. (Use a whole page for the diagram if desired).

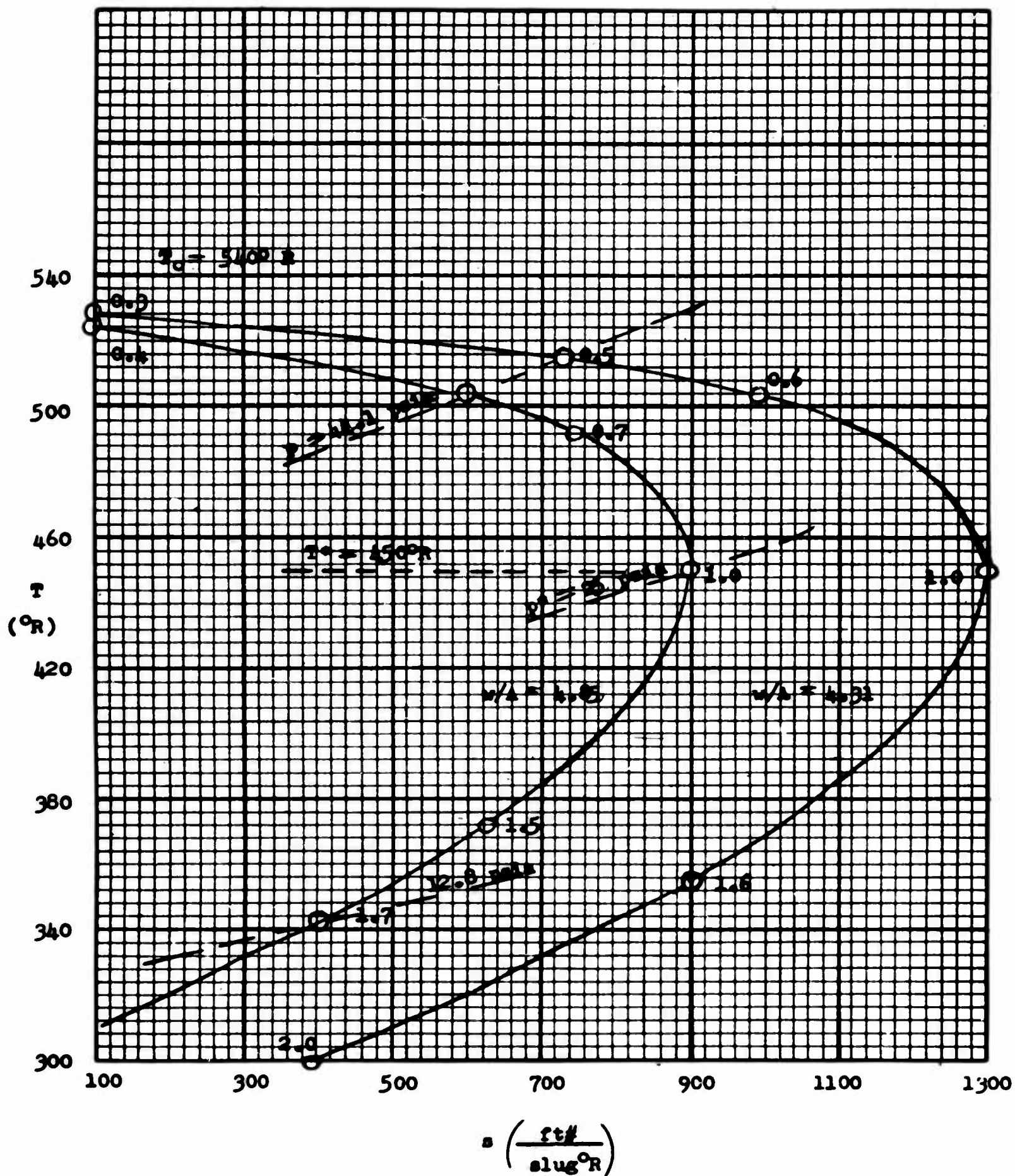
- 13.1 (a) A wedge with included angle of  $20^\circ$  is placed in a flow which  $p_0 = 100$  psia,  $p = 20$  psia, and  $T_0 = 80^\circ$  F. What are the exit stream properties from the attached plane shock that results, i.e.,  $p$ ,  $T$ ,  $M$ ,  $V$ ,  $p_0$ , and  $T_0$ ?
- (b) What is the maximum  $p$  possible in this flow for an attached shock wave to occur?
- (c) What is the maximum included wedge angle for an attached shock to occur in the flow of (a)?

- 14.1 Assume a flow to originate in a large reservoir with  $T = 540^\circ \text{R}$ ,  $p = 75.1 \text{ psia}$ , and, arbitrarily,  $s = 100 \frac{\text{ft} \cdot \text{lb}}{\text{slug}^\circ \text{R}}$ . Let the flow proceed through a convergent nozzle to  $M_1 = 0.4$ , thence through a simple frictional duct. Plot the locus of fluid states (Fanno line). In the computations let  $T = 503, 492, 450, 372$ , and  $342^\circ \text{R}$ , respectively. (Note:  $\frac{W}{A} = \rho_1 V_1$ ).
- 14.2 To cause the flow of 12.1 to proceed from  $M = 0.4$  to  $M = 1.0$  requires a certain duct length and exhaust region pressure  $\leq p^*$  of the Fanno line. Suppose these conditions are met such that  $M = 1$  at exit of duct and then the duct length is increased, everything else remaining fixed. Sketch on the  $T$ - $s$  chart of problem 14.1 the new flow process.

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P14.2

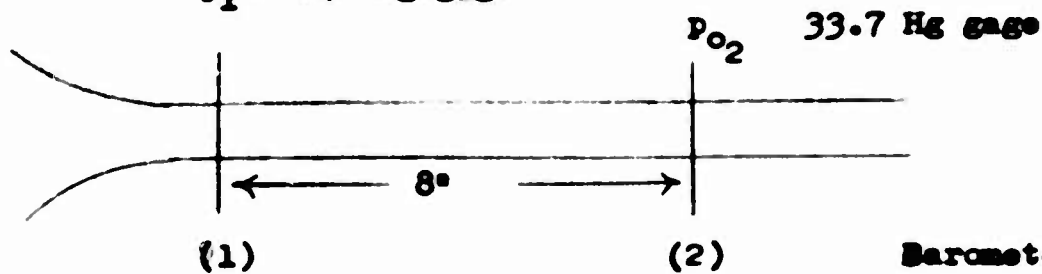
FANNO LINES FOR PERFECT GAS -  $T_0 = 540^\circ \text{R}$  PARAMETER IS  $w/A$  IN SLUGS/SEC-  
 $\text{FT}^2$  NUMBER BY CIRCLE O SHOWS MACH NO. AT THAT POINT



15.1  $T_{o1} = 520^\circ \text{ R}$

$p_{o1} = 40^\circ \text{ Hg gage}$

$p_1 = 24^\circ \text{ Hg gage}$



Barometer =  $29.5^\circ \text{ Hg}$

Tube diameter =  $0.760^\circ$

Find friction factor  $f$ .

Ans.  $f = 0.0067$

15.2 What is force of fluid on pipe of (15.1) between sections (1) and (2)? Ans.  $\bar{F} = 1.1\#$

15.3 Air is flowing through a circular pipe of  $1\#$  diameter with following inlet conditions.

$p_{o1} = 18 \text{ psia}$

friction factor  $f = 0.0025$

$p_1 = 15.18 \text{ psia}$

$p_{o1} = 600^\circ \text{ R}$

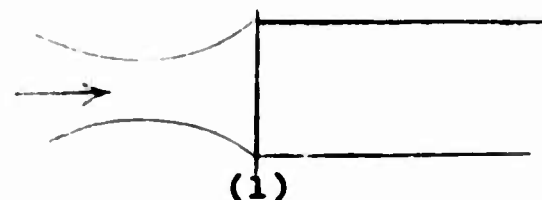
- If Mach equals one at pipe exit, what is pipe length and maximum discharge region pressure? ( $L_{\max} = 107^\circ$   $p = 7.11 \text{ psia}$ )
- If pipe is  $60^\circ$  long what discharge region pressure is required to maintain above inlet conditions? ( $12.38 \text{ psia}$ )
- With pipe length =  $60^\circ$  and back pressure =  $7.11 \text{ psia}$  what is  $w/A$ ? ( $1.44 \text{ slugs/sec-ft}^2$ .)
- If after condition (b) is established, the pipe length is increased to  $107^\circ$  what is  $(w/A)$ ? ( $1.13 \text{ slugs/sec-ft}^2$ .)

15.4 Consider the supersonic flow of air in a  $1^\circ$  diameter pipe with  $f = 0.0025$  and with following inlet conditions

$p_{o1} = 20 \text{ psia}$

$p_1 = 2.55 \text{ psia}$

$T_{o1} = 600^\circ \text{ R}$

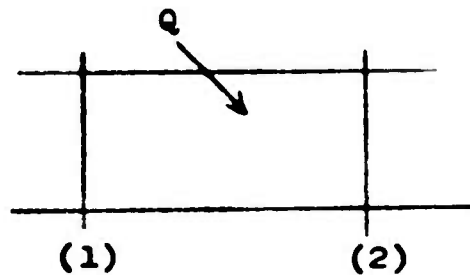


- a. What pipe length and exhaust pressure will give  $M = 1$  at pipe exit? ( $L_{\max} = 30.5''$   $p = 6.25$  psia)
- b. As the pipe length is increased beyond the value determined above and the exhaust pressure remains constant will the pipe inlet conditions change immediately or only after a certain pipe length has been reached? Explain.
- c. What pipe length is required to produce a shock in the flow at the point corresponding to  $M = 1.7$  (i.e. the shock inlet Mach number,  $M_x$  equals 1.71)? (45.4'')
- d. Compute  $W_a$  and  $W_c$ .
- e. What is maximum pipe length for sonic flow in nozzle throat and at pipe exit and subsonic flow elsewhere in system? (293'')

17.1 For flow with simple  $T_o$  effects show that

$$\frac{dp_o}{dT_o} < 0 \quad \left( \frac{dp}{dT_o} \right) \begin{matrix} < 0 & \text{for } M < 1 \\ > 0 & \text{for } M > 1 \end{matrix}$$

17.2 Air is flowing through a frictionless constant area duct with 600 Btu of heat added per slug of air between stations (1) and (2). For the conditions



$$\begin{aligned} p_1 &= 75 \text{ psia} \\ p_{o1} &= 100 \text{ psia} \\ T_{o1} &= 560^\circ \text{ R} \end{aligned}$$

given with figure determine  $p_{o2}$  and  $\frac{M}{A}$ . (See problem 7.7 for answers)

17.3 A ramjet combustor of constant area is operating cold (no fuel flow) with following inlet conditions

$$\begin{aligned} p_o &= 21.38 \text{ psia} & T_o &= 500^\circ \text{ R} \\ p_1 &= 18 \text{ psia} & A_1 &= 1.0 \text{ ft}^2 \end{aligned}$$

- Assuming frictionless flow what is maximum permissible heating between inlet and exit without affecting inlet conditions? ( $T_o = 725^\circ \text{ R}$ )
- If 774 Btu per slug of air are added between (1) and (2) what are the combustor exit conditions and mass rate of flow?
- Assume combustion gives  $T_{o2} = 2000^\circ \text{ R}$ . What are the new inlet conditions and new mass rate of flow?

- 18.1 A convergent-divergent diffuser is to be designed for a Mach Number of 2. Assuming no friction, compare the efficiency and the percent loss in stagnation pressure for the following cases:
- (a) The best possible design is employed.
  - (b) The design is conservative, with a throat area 5% larger than that required for starting, and with the shock located during operation at an area 5% greater than the throat area.
  - (c) The converging portion is eliminated, and the process comprises a normal shock followed by reversible subsonic compression.
- 18.2 A ram jet aircraft is to fly at 40,000 ft altitude with a speed of 2000 mph.
- (a) Design the best convergent-divergent diffuser for this aircraft, and compute for it the efficiency and the percent loss in stagnation pressure.
  - (b) Suppose that it were possible to overspeed the aircraft to 2400 mph. Design the best convergent-divergent diffuser which could then be used, and find for it the efficiency and the percent loss in stagnation pressure at the operating speed of 2000 mph.
  - (c) Below what flight speed will the shock always be disorged.
- 18.3 A supersonic wind tunnel is to be designed for a Mach Number of two with a test section one sq ft in area. The general arrangement of the tunnel will be as follows:

Air will be taken from the atmosphere (14.7 psia, 70°F) and will be accelerated to a Mach No. of 2 in a converging-diverging nozzle. From the test section the air will be diffused to substantially zero velocity in a diffuser and will then be discharged to the atmosphere by a compressor.

The design of the tunnel and compressor will be based on the following assumptions: (1) The nozzle is frictionless to the throat, while the overall efficiency ( $\Delta h/\Delta h_s$ ) of the nozzle, from entrance to exit, is 95%. (2) Although an attempt will be made to diffuse the supersonic stream through a throat, for purposes of design the assumption will be made that the stream with  $M = 2$  passes through a normal shock and that the subsonic stream is then diffused with an efficiency ( $\Delta h_s/\Delta h$ ) of 80%. The conservative nature of this assumption will tend to balance the fact that no account is taken of losses in the test section. (3) The compressor has an efficiency, ( $\Delta h_s/\Delta h$ ) based on the reversible adiabatic work of compression of 82%.

- (a) Make a sketch, to scale, of this tunnel, assuming that passageways are round in cross-section.
- (b) Indicate on the sketch the pressure (psia) and temperature (°F) at the test section, at the entrance to the compressor, and at the exit of the compressor.
- (c) Specify the diameter of the nozzle throat.
- (d) Specify, for the compressor, the pressure ratio, the volume rate of flow at inlet (cfm), and the horsepower required for operation.



- 19.1 Suppose that a blast wave which might have been initiated by an atomic bomb explosion is traveling through air at standard atmospheric conditions with a speed of 200,000 ft/sec.

Estimate the changes in pressure (atm), temperature ( $T$ ), stagnation pressure (atm), stagnation temperature ( $T$ ), and velocity (ft/sec) produced by the wave with respect to an observer who is stationary with respect to the undisturbed air.